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GAUSS CONGRUENCES IN ALGEBRAIC NUMBER FIELDS

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Abstract. In this miniature note we generalize the classical Gauss congruences for integers to rings of integers in algebraic number fields.

Recall that the classical Gauss congruence for integers states that, for $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, the following identity holds true:

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) a^d \equiv 0 \pmod{n},$$

where $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^m, & \text{if } n \text{ is a product of } m \text{ different primes}, \\ 0, & \text{otherwise.} \end{cases}$$

The abovestated identity generalizes in a surprisingly easy and natural way to rings of integers in algebraic function fields.

Let K be an algebraic number field and denote by $\mathcal{O}_K$ its ring of integers. Denote by $\mathcal{I}(\mathcal{O}_K)$ the family of all ideals of $\mathcal{O}_K$ and by $\operatorname{Spec} \mathcal{O}_K$ its prime

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spectrum. Further, denote by $N\colon \mathcal{I}(\mathcal{O}_K) \to \mathbb{N}$ the absolute norm function defined by the size of the (necessarily finite) quotient ring:

$$N(\mathfrak{n}) = |\mathcal{O}_K/\mathfrak{n}|.$$

Here and later on, for $a, b \in \mathcal{O}_K$ and $\mathfrak{n} \in \mathcal{I}(\mathcal{O}_K)$, by $a \equiv b (\operatorname{mod} \mathfrak{n})$ we shall understand $a - b \in \mathfrak{n}$.

As $\mathcal{O}_K$ is a Dedeking domain, every nonzero ideal $\mathfrak{n}$ of $\mathcal{O}_K$ can be uniquely represented as a product of prime ideals of $\mathcal{O}_K$, so that one can consider the following generalization of the Möbius fuction, which is due to Shapiro ([1]):

$$\mu(\mathfrak{n}) = \begin{cases} 1, & \text{if } \mathfrak{n} = 0, \\ (-1)^m, & \text{if } \mathfrak{n} \text{ is a product of } m \text{ different prime ideals,} \\ 0, & \text{otherwise.} \end{cases}$$

With this definition of the function $\mu\colon \mathcal{I}(\mathcal{O}_K) \to \{-1, 0, 1\}$, we shall prove the following version of the Gauss identity for number fields:

Theorem 1. *Let $a \in \mathcal{O}_K$, $\mathfrak{n} \in \mathcal{I}(\mathcal{O}_K)$. Then*

$$\sum_{\mathfrak{d}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{d}}\right) a^{N(\mathfrak{d})} \equiv 0 \ (\operatorname{mod} \mathfrak{n}).$$

For the proof we will use a version of Euler's Theorem for number fields. We shall state it here together with a proof for the sake of the completeness of our exposition, however there is no claim to its originality whatsoever.

Proposition 2 (Euler's Theorem). *Let $a \in \mathcal{O}_K$, $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ and $k \in \mathbb{N}$. Then*

$$a^{N(\mathfrak{p})^k} \equiv a^{N(\mathfrak{p})^{k-1}} \ (\operatorname{mod} \mathfrak{p}^k).$$

Proof. One needs to evaluate the number of units in the ring $\mathcal{O}_K/\mathfrak{p}^k$. The canonical map $\mathcal{O}_K/\mathfrak{p}^k \to \mathcal{O}_K/\mathfrak{p}$ given by $x + \mathfrak{p}^k \mapsto x + \mathfrak{p}$ is a well-defined ring homomorphism whose kernel is equal to $\mathfrak{p}/\mathfrak{p}^k$. As $\mathcal{O}_K$ is a Dedekind domain, the prime ideal $\mathfrak{p}$ is also maximal and hence $\mathcal{O}_K/\mathfrak{p}$ is a field, so that the ideal $\mathfrak{p}/\mathfrak{p}^k$ is maximal. Since $\sqrt{\mathfrak{p}^k} = \sqrt{\mathfrak{p}} = \mathfrak{p}$ is a maximal ideal, $\mathcal{O}_K/\mathfrak{p}^k$ is local, and thus $\mathfrak{p}/\mathfrak{p}^k$ is equal precisely to the set of non-units of $\mathcal{O}_K/\mathfrak{p}^k$. Considering the chain of additive Abelian groups $\mathfrak{p}^k \subseteq \mathfrak{p}^{k-1} \subseteq \ldots \mathfrak{p}^2 \subseteq \mathfrak{p}$ and using the isomorphism theorem combined with the Lagrange theorem, we get

$$|\mathfrak{p}/\mathfrak{p}^k| = (\mathfrak{p} : \mathfrak{p}^2) \cdot (\mathfrak{p}^2 : \mathfrak{p}^3) \cdot \ldots \cdot (\mathfrak{p}^{k-1} : \mathfrak{p}^k).$$

Each quotient group $\mathfrak{p}^i/\mathfrak{p}^{i+1}$, $i \in \{1, \dots, k-1\}$, has a structure of a $\mathcal{O}_K/\mathfrak{p}$-vector space, and its dimension is equal to 1. Indeed, let $x \in \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$ and $\mathfrak{a} = (x) + \mathfrak{p}^{i+1}$. Then $\mathfrak{p}^i \supseteq \mathfrak{a} \supsetneq \mathfrak{p}^{i+1}$, and, consequently, $\mathfrak{a} = \mathfrak{p}^i$, for otherwise $\frac{\mathfrak{a}}{\mathfrak{p}^i}$ would be a proper divisor of $\mathfrak{p} = \frac{\mathfrak{p}^{i+1}}{\mathfrak{p}^i}$. Hence $x + \mathfrak{p}^{i+1}$ is a basis of the $\mathcal{O}_K/\mathfrak{p}$-vector space $\mathfrak{p}^i/\mathfrak{p}^{i+1}$.

Therefore the number of units of the ring $\mathcal{O}_K/\mathfrak{p}^k$ is equal to:

$$|\mathcal{O}_K/\mathfrak{p}^k| - |\mathfrak{p}/\mathfrak{p}^k| = N(\mathfrak{p}^k) - |\mathcal{O}_K/\mathfrak{p}|^{k-1} = N(\mathfrak{p}^k) - N(\mathfrak{p})^{k-1}.$$

As the absolute norm is multiplicative, $N(\mathfrak{p}^k) = N(\mathfrak{p})^k$ and hence

$$(a + \mathfrak{p}^k)^{N(\mathfrak{p})^k - N(\mathfrak{p})^{k-1}} = a^{N(\mathfrak{p})^k - N(\mathfrak{p})^{k-1}} + \mathfrak{p}^k = 1 + \mathfrak{p}^k,$$

or, equivalently, $a^{N(\mathfrak{p})^k} \equiv a^{N(\mathfrak{p})^{k-1}} (\operatorname{mod} \mathfrak{p}^k)$. □

We can now proceed to the proof of Theorem 1:

Proof. Fix $a \in \mathcal{O}_K$ and $\mathfrak{n} \in \mathcal{I}(\mathcal{O}_K)$. Let $\mathfrak{n} = \mathfrak{p}_1^{k_1} \cdot \ldots \cdot \mathfrak{p}_m^{k_m}$ be the unique factorization of $\mathfrak{n}$ into a product of prime ideals. By the definition of the function μ, the set of divisors of $\mathfrak{n}$ whose value of μ is nonzero is equal to:

$$\{\mathfrak{p}_{j_1} \cdot \ldots \cdot \mathfrak{p}_{j_l} \mid 1 \leqslant j_1 < \ldots < j_l \leqslant m, l \in \{0, \dots, m\}\},$$

where by product of 0 ideals we understand the zero ideal 0. Thus

$$\begin{aligned}
\sum_{\mathfrak{d}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{d}}\right) a^{N(\mathfrak{d})} &= \sum_{l=0}^{m} \sum_{1\leqslant j_1<\ldots<j_l\leqslant m} (-1)^l a^{N\left(\frac{\mathfrak{p}_1^{k_1}\cdot\ldots\cdot\mathfrak{p}_m^{k_m}}{\mathfrak{p}_{j_1}\cdot\ldots\cdot\mathfrak{p}_{j_l}}\right)} \\
&= \sum_{l=0}^{m} \sum_{1\leqslant j_1<\ldots<j_l\leqslant m} (-1)^l a^{\frac{N(\mathfrak{p}_1)^{k_1}\cdot\ldots\cdot N(\mathfrak{p}_m)^{k_m}}{N(\mathfrak{p}_{j_1})\cdot\ldots\cdot N(\mathfrak{p}_{j_l})}} \\
&= \sum_{l=0}^{m-1} \sum_{2\leqslant j_1<\ldots<j_l\leqslant m} \left[(-1)^l a^{N(\mathfrak{p}_1)^{k_1}\frac{N(\mathfrak{p}_2)^{k_2}\cdot\ldots\cdot N(\mathfrak{p}_m)^{k_m}}{N(\mathfrak{p}_{j_1})\cdot\ldots\cdot N(\mathfrak{p}_{j_l})}}\right. \\
&\quad \left. -(-1)^l a^{N(\mathfrak{p}_1)^{k_1-1}\frac{N(\mathfrak{p}_2)^{k_2}\cdot\ldots\cdot N(\mathfrak{p}_m)^{k_m}}{N(\mathfrak{p}_{j_1})\cdot\ldots\cdot N(\mathfrak{p}_{j_l})}}\right].
\end{aligned}$$

By Proposition 2, $a^{N(\mathfrak{p}_1)^{k_1}} \equiv a^{N(\mathfrak{p}_1)^{k_1-1}} (\operatorname{mod} \mathfrak{p}_1^{k_1})$. Consequently,

$$(-1)^l a^{N(\mathfrak{p}_1)^{k_1} \frac{N(\mathfrak{p}_2)^{k_2} \cdot \ldots \cdot N(\mathfrak{p}_m)^{k_m}}{N(\mathfrak{p}_{j_1}) \cdot \ldots \cdot N(\mathfrak{p}_{j_l})}} \equiv (-1)^l a^{N(\mathfrak{p}_1)^{k_1-1} \frac{N(\mathfrak{p}_2)^{k_2} \cdot \ldots \cdot N(\mathfrak{p}_m)^{k_m}}{N(\mathfrak{p}_{j_1}) \cdot \ldots \cdot N(\mathfrak{p}_{j_l})}} (\operatorname{mod} \mathfrak{p}_1^{k_1}),$$

for $2 \leqslant j_1 < \ldots < j_l \leqslant m$, $l \in \{0, \ldots, m-1\}$, and hence

$$\sum_{\mathfrak{d} | \mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{d}}\right) a^{N(\mathfrak{d})} \equiv 0 \; (\operatorname{mod} \mathfrak{p}_1^{k_1}).$$

Repeating the argument for the ideals $\mathfrak{p}_2, \ldots, \mathfrak{p}_m$ we get

$$\sum_{\mathfrak{d} | \mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{d}}\right) a^{N(\mathfrak{d})} \equiv 0 \; (\operatorname{mod} \mathfrak{p}_i^{k_i}),$$

for $i \in \{1, \ldots, m\}$, so that

$$\sum_{\mathfrak{d} | \mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{d}}\right) a^{N(\mathfrak{d})} \equiv 0 \; (\operatorname{mod} \mathfrak{n}). \qquad \square$$

REMARK 3. We note that taking $K = \mathbb{Q}$ with $\mathcal{O}_K = \mathbb{Z}$ Theorem 1 yields the classical version of the Gauss congruence.

References

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