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A PARAMETRIC FUNCTIONAL EQUATION ORIGINATING FROM NUMBER THEORY

Aziz Mouzoun, Driss Zeglami, Youssef Aissi

Abstract. Let S be a semigroup and $\alpha, \beta \in \mathbb{R}$. The purpose of this paper is to determine the general solution $f\colon \mathbb{R}^2 \to S$ of the following parametric functional equation

$$f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1 + \beta y_1y_2) = f(x_1, y_1)f(x_2, y_2),$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, that generalizes some functional equations arising from number theory and is connected with the characterizations of the determinant of matrices.

1. Introduction

Throughout this paper S denotes a semigroup (i.e., a non-empty set equipped with an associative composition rule $(x, y) \to xy$), $\mathbb{K}$ denotes either the set of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, and $\alpha, \beta \in \mathbb{R}$. The semigroup S will represent the range space of the solutions in the second section of this paper. We equip $\mathbb{R}^2$ with the multiplication rule $*_{\alpha,\beta}$ defined by

$$(x_1, y_1) *_{\alpha,\beta} (x_2, y_2) = (x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1 + \beta y_1y_2),$$

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for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. The rule makes $\mathbb{R}^2$ into an abelian monoid with neutral element $(1, 0)$.

We introduce the multiplicative Cauchy $*_{\alpha,\beta}$-functional equation

$$(E(\alpha,\beta)) \qquad f((x_1, y_1) *_{\alpha,\beta} (x_2, y_2)) = f(x_1, y_1)f(x_2, y_2),$$

i.e.

$$f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1 + \beta y_1y_2) = f(x_1, y_1)f(x_2, y_2), \tag{1.1}$$

where $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, α, β are fixed real parameters and $f: (\mathbb{R}^2, *_{\alpha,\beta}) \to S$ is the unknown multiplicative function to be determined.

Let us mention some recent contributions to the theory of functional equations related to (1.1). For $\beta = 0$, where $(E(\alpha, \beta))$ reduces to $(E(\alpha, 0))$, Berrone and Dieulefait ([5]) characterized the solution $f: \mathbb{R}^2 \to \mathbb{R}$ of the functional equation

$$f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1) = f(x_1, y_1)f(x_2, y_2),$$

that arises from the product of two numbers in a *quadratic number field.* Functional equations which result from the formula of the product of two numbers in a pure cubic (resp. quartic) number field were investigated in [11] (resp. [15]). Another particular instance of (1.1) is the functional equation

$$f(x_1x_2 - y_1y_2, x_1y_2 + x_2y_1 + y_1y_2) = f(x_1, y_1)f(x_2, y_2), \tag{1.2}$$

which was derived from the *Proth identity.* Ebanks ([8]) found the solutions $f: \mathbb{F}^2 \to S$ of (1.2), here $\mathbb{F}$ is any field containing $\mathbb{Q}(i\sqrt{3})$ and S is a commutative semigroup, and Chavez and Sahoo ([6]) determined its solutions $f: \mathbb{K}^2 \to \mathbb{K}$. In [9] Jung and Bae discussed the form of the solutions $f: \mathbb{R}^2 \to \mathbb{R}$ of

$$f(x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = f(x_1, y_1)f(x_2, y_2),$$

which arises from the following identity $(x_1x_2 + y_1y_2)^2 + (x_1y_2 - x_2y_1)^2 = (x_1^2 + y_1^2)\left(x_2^2 + y_2^2\right)$. Akkouchi and Rhali ([4]), Chavez and Sahoo ([6]) described, for a fixed $\lambda \in \mathbb{K}^* := \mathbb{K}\setminus\{0\}$, the solutions $f: \mathbb{K}^2 \to \mathbb{K}$ and $f: \mathbb{K}^2 \to S$, respectively, of the functional equation

$$f(x_1x_2 + (\lambda - 1)y_1y_2, x_1y_2 + x_2y_1 + (\lambda - 2)y_1y_2) = f(x_1, y_1)f(x_2, y_2),$$

which is connected to the determinant of some matrices.

Recently, the authors ([10]) treated another kind of equation than (1.1). They described the solutions $f\colon \mathbb{R}^2 \to M_2(\mathbb{C})$ of the matrix functional equation

$$(1.3) \qquad f(x_1x_2 + \alpha y_1y_2, x_1y_2 + \gamma x_2y_1) = f(x_1, y_1)f(x_2, y_2),$$

where α, γ are fixed real numbers. Of course, Eq. (1.3) differs from (1.1) when $\gamma \neq 1$.

In connection with the characterization of functional equations arising from the number theory, the present paper complements and contains the existing results by finding the solutions $f\colon \mathbb{R}^2 \to S$ of the parametric functional equation $(E(\alpha, \beta))$. We impose no conditions like continuity on the solutions.

(1) We characterize, in terms of multiplicative functions from $(\mathbb{R}, \cdot)$ or $(\mathbb{C}, \cdot)$ to S, the solutions $f\colon \mathbb{R}^2 \to S$ of $(E(\alpha, \beta))$.
(2) We find explicit expressions for the functions $f\colon \mathbb{R}^2 \to \mathbb{C}$ satisfying the equation $(E(\alpha, \beta))$, and
(3) we describe, in terms of multiplicative functions $M\colon (\mathbb{R}, \cdot) \to \mathbb{R}$ and additive ones $A\colon (\mathbb{R}, +) \to \mathbb{R}$, its real-valued solutions.
(4) By a more direct approach, we solve the particular instance of $(E(\alpha, \beta))$ for $\beta^2 + 4\alpha \neq 0$, in which $S = M_2(\mathbb{C})$.

Notation. Throughout this paper $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ with $\mathbb{K}^* = \mathbb{K}\backslash\{0\}$, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$, and S denotes a semigroup. That S is a regular semigroup means that for all $x \in S$ there exist $a \in S$ such that $x = xax$.

In the sequel, all semigroups and groups will be denoted using multiplicative notation. Let S_1, S_2 be semigroups. A function $\phi\colon S_1 \to S_2$ is said to be a semigroup morphism if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in S_1$. If the semigroup operation in S_2 is a multiplication, then the semigroup morphism ϕ is said to be a multiplicative function. If the semigroup operation in S_2 is the addition, then the semigroup morphism ϕ is said to be an additive function. A character on a group G is a multiplicative function $\chi\colon G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. As well known, any non-zero multiplicative function on a group is a character (see [13, Lemma 3.4(a)]). It is possible for a multiplicative function on S to take the value 0 on a proper non-empty subset of S. For any multiplicative function $\phi\colon S \to \mathbb{C}$ we use the notation

$$I_\phi := \{x \in S \mid \phi(x) = 0\}.$$

2. Main results

Inspired by papers [6, 8, 7], we will describe the solutions $f\colon \mathbb{R}^2 \to S$ of the functional equation $(E(\alpha,\beta))$. Let $\mathbb{H}$ be the set defined by

$$\mathbb{H} := \{(z,\bar{z}) \mid z \in \mathbb{C}\}.$$

We equip $\mathbb{H}$ with the multiplication rule $\diamond$ defined by

$$(z_1,\bar{z}_1) \diamond (z_2,\bar{z}_2) = (z_1 z_2, \overline{z_1 z_2}) \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

The following lemma presents a result that is essential for the proof of our main results.

Lemma 2.1. *Let $\alpha,\beta \in \mathbb{R}$ such that $\beta^2+4\alpha<0$. The map $\tau\colon (\mathbb{R}^2, *_{\alpha,\beta}) \to (\mathbb{H},\diamond)$ defined by*

$$\tau(x,y) = \Big(x + \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})y, x + \frac{1}{2}(\beta - i\sqrt{-\beta^2-4\alpha})y\Big), \quad x,y \in \mathbb{R},$$

is a bijective homomorphism.

Proof. With the notation $\xi := \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})$, we have

$$\begin{aligned}\tau((x_1,y_1) *_{\alpha,\beta} (x_2,y_2)) &= \tau(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1 + \beta y_1y_2)\\ &= ((x_1+\xi y_1)(x_2+\xi y_2), (x_1+\bar{\xi}y_1)(x_2+\bar{\xi}y_2))\\ &= (\tau_1(x_1,y_1)\tau_1(x_2,y_2), \tau_2(x_1,y_1)\tau_2(x_2,y_2))\\ &= \tau(x_1,y_1) \diamond \tau(x_2,y_2),\end{aligned}$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$. This implies that τ is an homomorphism from $(\mathbb{R}^2, *_{\alpha,\beta})$ to $(\mathbb{H},\diamond)$. To show that τ is bijective, it is elementary to see, for all $(z,\bar{z}) \in \mathbb{H}$ with $z = a+ib$, $(a,b) \in \mathbb{R}^2$, that

$$(x,y) = \left(a - \frac{\beta}{\sqrt{-\beta^2-4\alpha}}b, \frac{2}{\sqrt{-\beta^2-4\alpha}}b\right)$$

is the unique element of $\mathbb{R}^2$ such that $\tau(x,y) = (z,\bar{z})$. $\square$

The following theorem lists the solutions $f\colon \mathbb{R}^2 \to S$ of the equation $(E(\alpha,\beta))$ when $\beta^2+4\alpha \neq 0$.

THEOREM 2.2. *Let $\alpha,\beta \in \mathbb{R}$ such that $\beta^2+4\alpha \neq 0$. The general solution $f\colon \mathbb{R}^2 \to S$ of $(E(\alpha,\beta))$ depends on the sign of $\beta^2+4\alpha$ and is given by:*

(1) *If $\beta^2+4\alpha>0$, then*

$$f(x,y)=M_1\Big(x+\frac{1}{2}(\beta-\sqrt{\beta^2+4\alpha})y\Big)M_2\Big(x+\frac{1}{2}(\beta+\sqrt{\beta^2+4\alpha})y\Big),$$

for all $x,y\in\mathbb{R}$, where $M_1,M_2\colon(\mathbb{R},\cdot)\to S$ are multiplicative functions.

(2) *If $\beta^2+4\alpha<0$, then*

$$f(x,y)=M\Big(x+\frac{1}{2}(\beta+i\sqrt{-\beta^2-4\alpha})y\Big),$$

for all $x,y\in\mathbb{R}$, where $M\colon(\mathbb{C},\cdot)\to S$ is a multiplicative function.

PROOF. Let $f\colon\mathbb{R}^2\to S$ be a solution of $(E(\alpha,\beta))$. In solving equation $(E(\alpha,\beta))$, two different cases arise depending on the sign of $\beta^2+4\alpha$.

Case 1: If $\beta^2+4\alpha>0$, we distinguish between two subcases.

Subcase 1: Suppose first that $\alpha\neq 0$. Putting $\gamma=\sqrt{\beta^2+4\alpha}$, $s=\beta+\gamma$ and $\delta=\beta-\gamma$, it is easy to see that $s\delta=-4\alpha$, $s\neq 0$ and $\delta\neq 0$. We adopt the ideas of [6] to the situation at hand. In matrix terminology, $(E(\alpha,\beta))$ can be written as

$$\begin{aligned}(x_1,y_1)*_{\alpha,\beta}(x_2,y_2)&=(x_1x_2+\alpha y_1y_2,x_1y_2+x_2y_1+\beta y_1y_2)\\&=\begin{pmatrix}x_2 & \alpha y_2\\ y_2 & x_2+\beta y_2\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix},\end{aligned}$$

where $x_1,x_2,y_1,y_2\in\mathbb{R}$. The diagonalization of last equality gives us

$$(x_1,y_1)*_{\alpha,\beta}(x_2,y_2)=Q\begin{pmatrix}x_2+\frac{1}{2}\delta y_2 & 0\\ 0 & x_2+\frac{1}{2}sy_2\end{pmatrix}Q^{-1}\begin{pmatrix}x_1\\ y_1\end{pmatrix},$$

where $Q=\begin{pmatrix}1 & 1\\ -\frac{2}{s} & -\frac{2}{\delta}\end{pmatrix}$ and $Q^{-1}=\frac{\alpha}{\gamma}\begin{pmatrix}-\frac{2}{\delta} & -1\\ \frac{2}{s} & 1\end{pmatrix}$.

Hence, the equation $(E(\alpha,\beta))$ can be reformulated as

$$f\left(Q\begin{pmatrix} x_2+\frac{1}{2}\delta y_2 & 0 \\ 0 & x_2+\frac{1}{2}sy_2\end{pmatrix}Q^{-1}\begin{pmatrix} x_1 \\ y_1\end{pmatrix}\right)=f(x_1,y_1)f(x_2,y_2), \tag{2.1}$$

where $x_1,x_2,y_1,y_2\in\mathbb{R}$. We define the function: $\phi\colon \mathbb{R}^2\to S$ by

$$\phi(X):=f(QX),\quad X\in\mathbb{R}^2. \tag{2.2}$$

We use (2.2) to rewrite (2.1) in terms of ϕ as

$$\begin{aligned}\phi\left(\begin{pmatrix} x_2+\frac{1}{2}\delta y_2 & 0 \\ 0 & x_2+\frac{1}{2}sy_2\end{pmatrix}Q^{-1}\begin{pmatrix} x_1 \\ y_1\end{pmatrix}\right)\\ =\phi\left(Q^{-1}\begin{pmatrix} x_1 \\ y_1\end{pmatrix}\right)\phi(Q^{-1}\begin{pmatrix} x_2 \\ y_2\end{pmatrix}\right),\quad x_1,x_2,y_1,y_2\in\mathbb{R}.\end{aligned} \tag{2.3}$$

We make the change of variables

$$\begin{pmatrix} u_j \\ v_j\end{pmatrix}=Q^{-1}\begin{pmatrix} x_j \\ y_j\end{pmatrix}\quad \text{for } j=1,2. \tag{2.4}$$

We obtain after some computations that

$$\begin{pmatrix} x_2+\frac{1}{2}\delta y_2 & 0 \\ 0 & x_2+\frac{1}{2}sy_2\end{pmatrix}=\begin{pmatrix} \frac{-\gamma\delta}{2\alpha}u_2 & 0 \\ 0 & \frac{\gamma s}{2\alpha}v_2\end{pmatrix}.$$

By the change of variables (2.4), the equation (2.3) becomes

$$\phi\left(\begin{pmatrix} \frac{-\gamma\delta}{2\alpha}u_2 & 0 \\ 0 & \frac{\gamma s}{2\alpha}v_2\end{pmatrix}\begin{pmatrix} u_1 \\ v_1\end{pmatrix}\right)=\phi\begin{pmatrix} u_1 \\ v_1\end{pmatrix}\phi\begin{pmatrix} u_2 \\ v_2\end{pmatrix},\quad u_1,u_2,v_1,v_2\in\mathbb{R}.$$

This yields that

$$\phi\Big(\frac{-\gamma\delta}{2\alpha}u_1u_2,\frac{\gamma s}{2\alpha}v_1v_2\Big)=\phi(u_1,v_1)\phi(u_2,v_2),\quad u_1,u_2,v_1,v_2\in\mathbb{R}. \tag{2.5}$$

Let $h\colon \mathbb{R}^2 \to S$ be a function defined by $h(u,v) := \phi\big(-\frac{2\alpha}{\gamma\delta}u, \frac{2\alpha}{\gamma s}v\big)$, where $(u,v) \in \mathbb{R}^2$. Since $\alpha \neq 0$ we get that

$$\phi(u,v) = h\Big(\frac{-\gamma\delta}{2\alpha}u, \frac{\gamma s}{2\alpha}v\Big), \quad (u,v) \in \mathbb{R}^2. \tag{2.6}$$

By using (2.6) in (2.5) we find that

$$h\Big(\frac{-\gamma\delta}{2\alpha}u_1\frac{-\gamma\delta}{2\alpha}u_2, \frac{\gamma s}{2\alpha}v_1\frac{\gamma s}{2\alpha}v_2\Big) = h\Big(\frac{-\gamma\delta}{2\alpha}u_1, \frac{\gamma s}{2\alpha}v_1\Big)h\Big(\frac{-\gamma\delta}{2\alpha}u_2, \frac{\gamma s}{2\alpha}v_2\Big).$$

This yields that

$$h(x_1x_2, y_1y_2) = h(x_1,y_1)h(x_2,y_2), \quad x_1,y_1,x_2,y_2 \in \mathbb{R}. \tag{2.7}$$

If we put $y_1 = y_2 = 1$ and $x_1 = x_2 = 1$ separately in (2.7), we get respectively

$$h(x_1x_2, 1) = h(x_1,1)h(x_2,1), \quad x_1,x_2 \in \mathbb{R},$$
$$\text{and} \quad h(1, y_1y_2) = h(1,y_1)h(1,y_2), \quad y_1,y_2 \in \mathbb{R}.$$

These yield that there exist multiplicative functions $M_1, M_2\colon (\mathbb{R},\cdot) \to S$ such that $h(x,1) = M_1(x)$ and $h(1,y) = M_2(y)$ for all $x,y \in \mathbb{R}$. Since $h(x,y) = h(x,1)h(1,y)$ for all $x,y \in \mathbb{R}$, we deduce that $h(x,y) = M_1(x)M_2(y)$, $x,y \in \mathbb{R}$. So according to (2.6), we get

$$\phi(x,y) = M_1\Big(\frac{-\gamma\delta}{2\alpha}x\Big)M_2\Big(\frac{\gamma s}{2\alpha}y\Big). \tag{2.8}$$

From (2.2) and (2.8) we infer that

$$\begin{aligned}
f(x,y) &= \phi\Big(Q^{-1}\binom{x}{y}\Big)\\
&= \phi\Big(\frac{s}{2\gamma}x - \frac{\alpha}{\gamma}y, -\frac{\delta}{2\gamma}x + \frac{\alpha}{\gamma}y\Big)\\
&= M_1\Big(\frac{-\gamma\delta}{2\alpha}\Big(\frac{s}{2\gamma}x - \frac{\alpha}{\gamma}y\Big)\Big)M_2\Big(\frac{\gamma s}{2\alpha}\Big(-\frac{\delta}{2\gamma}x + \frac{\alpha}{\gamma}y\Big)\Big)\\
&= M_1\Big(x + \frac{\delta}{2}y\Big)M_2\Big(x + \frac{s}{2}y\Big)\\
&= M_1\Big(x + \frac{1}{2}(\beta - \sqrt{\beta^2 + 4\alpha})y\Big)M_2\Big(x + \frac{1}{2}(\beta + \sqrt{\beta^2 + 4\alpha})y\Big).
\end{aligned}$$

Subcase 2: If $\alpha = 0$, then $\beta \in \mathbb{R}^*$ and $(E(\alpha,\beta))$ becomes

$$f(x_1x_2, x_1y_2 + x_2y_1 + \beta y_1y_2) = f(x_1,y_1)f(x_2,y_2), \tag{2.9}$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Let $f\colon \mathbb{R}^2 \to S$ be a solution of (2.9). By using the function $\mathfrak{F}\colon \mathbb{R}^2 \to S$ defined by

$$\mathfrak{F}(u,v) := f(u, v/\beta), \quad u,v \in \mathbb{R}, \tag{2.10}$$

the equation (2.9) becomes

$$\mathfrak{F}(u_1,v_1)\mathfrak{F}(u_2,v_2) = \mathfrak{F}(u_1u_2, u_1v_2 + u_2v_1 + v_1v_2), \quad u_1,u_2,v_1,v_2 \in \mathbb{R}. \tag{2.11}$$

Let $k\colon \mathbb{R}^2 \to S$ be the function defined for any $x, y \in \mathbb{R}$ by

$$k(x,y) := \mathfrak{F}(x, y - x). \tag{2.12}$$

By using (2.12) in (2.11), we arrive at the functional equation

$$k(x_1,y_1)k(x_2,y_2) = k(x_1x_2, y_1y_2), \quad x_1,x_2,y_1,y_2 \in \mathbb{R}.$$

In a similar fashion as in (2.7), we deduce that there exist multiplicative functions $M_1, M_2\colon (\mathbb{R},\cdot) \to S$ such that

$$k(x,y) = M_1(x)M_2(y), \quad x,y \in \mathbb{R}. \tag{2.13}$$

Thus, by virtue of (2.12) and (2.10) in (2.13), we infer that

$$f(x,y) = M_1(x)M_2(x + \beta y), \quad x,y \in \mathbb{R}.$$

So we are in case (1) with $\alpha = 0$.

Case 2: Suppose that $\beta^2 + 4\alpha < 0$. We use the notations of Lemma 2.1. In term of the function $g\colon \mathbb{H} \to S$ defined by

$$g := f \circ \tau^{-1}, \tag{2.14}$$

the equation $(E(\alpha,\beta))$ reads as

$$g(\tau((x_1,y_1) *_{\alpha,\beta} (x_2,y_2))) = g(\tau(x_1,y_1))g(\tau(x_2,y_2)),$$

i.e.

$$g(\tau(x_1,y_1) \diamond \tau(x_2,y_2)) = g(\tau(x_1,y_1))g(\tau(x_2,y_2)).$$

For $\xi = \frac{1}{2}(\beta + i\sqrt{-\beta^2 - 4\alpha})$, the last equation becomes

$$g\left((x_1 + \xi y_1, x_1 + \bar{\xi} y_1) \diamond (x_2 + \xi y_2, x_2 + \bar{\xi} y_2)\right) = g\left(x_1 + \xi y_1, x_1 + \bar{\xi} y_1\right) g\left(x_2 + \xi y_2, x_2 + \bar{\xi} y_2\right). \tag{2.15}$$

If we put $z_i = x_i + \xi y_i$ for all $x_i, y_i \in \mathbb{R}$ and $i \in \{1, 2\}$ in (2.15), we get

$$g((z_1, \bar{z}_1) \diamond (z_2, \bar{z}_2)) = g(z_1, \bar{z}_1) g(z_2, \bar{z}_2), \quad z_1, z_2 \in \mathbb{C}.$$

This yields that

$$g(z_1 z_2, \bar{z}_1 \bar{z}_2) = g(z_1, \bar{z}_1) g(z_2, \bar{z}_2), \quad z_1, z_2 \in \mathbb{C},$$

which means that there exists a multiplicative function $M\colon (\mathbb{C}, \cdot) \to S$ such that $g(z, \bar{z}) = M(z)$, $z \in \mathbb{C}$. So from (2.14) we obtain

$$f(x, y) = M\Big(x + \frac{1}{2}(\beta + i\sqrt{-\beta^2 - 4\alpha})y\Big), \ x, y \in \mathbb{R}.$$

Hence we complete the proof of the first direction.

Conversely, simple computations prove that the formulas above for f define solutions of $(E(\alpha, \beta))$. □

- For $\mathbb{K} = \mathbb{R}$, as an immediate consequence of Theorem 2.2, taking $\beta = \lambda - 2$ and $\alpha = \lambda - 1$ where $\lambda \in \mathbb{R}^*$, we get [6, Theorem 3.3] on the semigroup-valued solutions of $(E(\alpha, \beta))$ on $\mathbb{R}^2$.
- As another interesting consequence of Theorem 2.2, on the solutions $f\colon \mathbb{R}^2 \to S$ of $(E(\alpha, \beta))$, we get [6, Theorem 3.2].

Now we focus on the solutions $f\colon \mathbb{R}^2 \to S$ of $(E(\alpha, \beta))$in the case $\beta^2 + 4\alpha = 0$.

Proposition 2.3. *Assume $\beta^2 + 4\alpha = 0$. If $f\colon \mathbb{R}^2 \to S$ is a solution of $(E(\alpha, \beta))$ then there exist multiplicative functions $M\colon (\mathbb{R}, \cdot) \to S$ and $\chi\colon (\mathbb{R}, +) \to S$ such that, for all $(x, y) \in \mathbb{R}^2$, we have*

(1) *for $\beta = 0$:*

$$f(x, y) = M(x)\chi\Big(\frac{y}{x}\Big) \ \ \text{if } x \neq 0 \ \ \text{and} \ \ f^2(0, y) = M(0),$$

(2) *for $\beta \neq 0$:*

$$f(x, y) = M\Big(x + \frac{\beta}{2}y\Big)\chi\Big(\frac{\beta y}{2x + \beta y}\Big) \ \ \text{if } x + \frac{\beta}{2}y \neq 0 \ \ \text{and} \ \ f^2\Big(-\frac{1}{2}\beta y, y\Big) = M(0).$$

Moreover, in both cases, if S is uniquely 2-divisible semigroup, then $f\big(-\frac{1}{2}\beta y, y\big) = M(0)$ *for all* $y \in \mathbb{R}$.

Proof. Let $f\colon \mathbb{R}^2 \to S$ be a solution of $(E(\alpha,\beta))$. Since $\beta^2 + 4\alpha = 0$, then $(E(\alpha,\beta))$ is $(E(-\frac{1}{4}\beta^2,\beta))$:

$$f\Big(x_1x_2 - \frac{\beta^2}{4}y_1y_2, x_1y_2 + x_2y_1 + \beta y_1y_2\Big) = f(x_1,y_1)f(x_2,y_2). \tag{2.16}$$

If $\beta = 0$, then (2.16) becomes

$$f(x_1x_2, x_1y_2 + x_2y_1) = f(x_1,y_1)f(x_2,y_2). \tag{2.17}$$

Putting $y_1 = y_2 = 0$ and $x_1 = x_2 = 1$ separately in (2.17), we obtain respectively

$$f(x_1x_2, 0) = f(x_1,0)f(x_2,0), \quad x_1,x_2 \in \mathbb{R},$$
$$\text{and} \quad f(1, y_1 + y_2) = f(1,y_1)f(1,y_2), \quad y_1,y_2 \in \mathbb{R}.$$

These yield that there exist multiplicative functions $M\colon (\mathbb{R},\cdot) \to S$ and $\chi\colon (\mathbb{R},+) \to S$ such that $f(x,0) =: M(x)$ and $f(1,x) =: \chi(x)$ for all $x \in \mathbb{R}$. If $x \neq 0$, then we have $f(x,y) = f(x,0)f(1,\frac{y}{x})$, which implies that

$$f(x,y) = M(x)\chi\Big(\frac{y}{x}\Big) \quad \text{for all } (x,y) \in \mathbb{R}^* \times \mathbb{R}.$$

Otherwise, we have $f^2(0,y) = f(0,0) = M(0)$. If we suppose that S is an uniquely 2-divisible semigroup, then we get $f(0,y) = M(0)$ for all $y \in \mathbb{R}$, because $M^2(0) = M(0)$.

Suppose now that $\beta \neq 0$. Let $F\colon \mathbb{R}^2 \to S$ be a function defined for any $u,v \in \mathbb{R}$, by

$$F(u,v) := f(u, 2v/\beta). \tag{2.18}$$

Then, the equation (2.16) can be expressed in terms of F as follows

$$\begin{aligned} F(u_1,v_1)F(u_2,v_2) &= f(u_1,2v_1/\beta)f(u_2,2v_2/\beta) \\ &= F\Big(u_1u_2 - \frac{\beta^2}{4}\frac{2v_1}{\beta}\frac{2v_2}{\beta}, \frac{\beta}{2}\Big(u_1\frac{2v_2}{\beta} + u_2\frac{2v_1}{\beta} + \beta\frac{2v_1}{\beta}\frac{2v_2}{\beta}\Big)\Big) \\ &= F(u_1u_2 - v_1v_2, u_1v_2 + u_2v_1 + 2v_1v_2), \quad u_1,u_2,v_1,v_2 \in \mathbb{R}. \end{aligned} \tag{2.19}$$

Define the function $g\colon \mathbb{R}^2 \to S$ for any $x, y \in \mathbb{R}$, by

$$g(x,y) := F(x-y,y). \tag{2.20}$$

By using (2.20) in (2.19), we arrive at the functional equation

$$g((x+y)(u+v),(x+y)v+y(u+v)) = g(x+y,y)g(u+v,v), \tag{2.21}$$

where $x, y, u, v \in \mathbb{R}$. If we set $x_1 = x+y,\ y_1 = y,\ x_2 = u+v$ and $y_2 = v$ in (2.21), we find that

$$g(x_1x_2, x_1y_2 + x_2y_1) = g(x_1,y_1)g(x_2,y_2),$$

i.e. g is a solution of (2.17). So in view of the previous discussions we have for all $x, y \in \mathbb{R}$

$$g(x,y) = M(x)\chi\Big(\frac{y}{x}\Big) \quad \text{if } x \neq 0 \quad \text{and} \quad g^2(0,y) = g(0,0), \tag{2.22}$$

where $M\colon (\mathbb{R},\cdot) \to S$ and $\chi\colon (\mathbb{R},+) \to S$ are multiplicative functions and $g(0,0) = M(0)$. From (2.20) and (2.22) we get

$$\begin{cases} F(x,y) = M(x+y)\chi\Big(\dfrac{y}{x+y}\Big), & x+y \neq 0,\\ F^2(x,y) = F(0,0), & x+y = 0, \quad x,y \in \mathbb{R}. \end{cases} \tag{2.23}$$

By using (2.23) in (2.18) we obtain $f(x,y) = M(x+\frac{\beta}{2}y)\chi\big(\frac{\beta y}{2x+\beta y}\big)$ if $x+\frac{\beta}{2}y \neq 0$ and $f^2(-\frac{1}{2}\beta y, y) = f(0,0) = M(0)$. If S is uniquely 2-divisible multiplicative semigroup, we get $f(-\frac{1}{2}\beta y, y) = M(0)$. □

3. The scalar solutions of $(E(\alpha,\beta))$

In this section, we describe the solutions $f\colon \mathbb{R}^2 \to \mathbb{K}$ of $(E(\alpha,\beta))$. The previous discussion allows us to determine the complex-valued solutions of the equation $(E(\alpha,\beta))$.

Corollary 3.1. *The general solution $f\colon \mathbb{R}^2 \to \mathbb{C}$ of $(E(\alpha,\beta))$ depends on the sign of $\beta^2 + 4\alpha$ and is given by:*

(1) *If $\beta^2 + 4\alpha = 0$, then either $f \equiv 1$ or there exist multiplicative functions $M\colon (\mathbb{R},\cdot) \to \mathbb{C}$ and $\chi\colon (\mathbb{R},+) \to \mathbb{C}$ such that for all $x, y \in \mathbb{R}$ we have*

(i) *For* $\beta = 0$,

$$f(x,y) = \begin{cases} M(x)\chi\Big(\dfrac{y}{x}\Big), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(ii) *For* $\beta \neq 0$,

$$f(x,y) = \begin{cases} M\Big(x + \dfrac{\beta}{2}y\Big)\chi\Big(\dfrac{\beta y}{2x + \beta y}\Big), & x + \dfrac{\beta}{2}y \neq 0, \\ 0, & \textit{else}. \end{cases}$$

(2) *If* $\beta^2 + 4\alpha > 0$, *then for all* $(x,y) \in \mathbb{R}^2$ *we have*

$$f(x,y) = M_1\Big(x + \frac{1}{2}(\beta - \sqrt{\beta^2 + 4\alpha})y\Big)M_2\Big(x + \frac{1}{2}(\beta + \sqrt{\beta^2 + 4\alpha})y\Big),$$

where $M_1, M_2 \colon (\mathbb{R}, \cdot) \to \mathbb{C}$ *are multiplicative functions.*

(3) *If* $\beta^2 + 4\alpha < 0$, *then for all* $(x,y) \in \mathbb{R}^2$ *we have*

$$f(x,y) = M\Big(x + \frac{1}{2}(\beta + i\sqrt{-\beta^2 - 4\alpha})y\Big),$$

where $M \colon (\mathbb{C}, \cdot) \to \mathbb{C}$ *is a multiplicative function.*

Proof. Let $f \colon \mathbb{R}^2 \to \mathbb{C}$ be a solution of $(E(\alpha,\beta))$. We have the following two cases:

Case 1: Suppose that $\beta^2 + 4\alpha = 0$. We distinguish between two subcases:

(i) if $\beta = 0$, then according to Proposition 2.3 we infer that there exist multiplicative functions $M \colon (\mathbb{R}^*, \cdot) \to \mathbb{C}$ and $\chi \colon (\mathbb{R}, +) \to \mathbb{C}$ such that $f(x,y) = M(x)\chi(\dfrac{y}{x})$ for all $(x,y) \in \mathbb{R}^* \times \mathbb{R}$ and $f^2(0,y) = f(0,0)$ for any $y \in \mathbb{R}$. Since $f^2(0,0) = f(0,0)$ then $f(0,0) = 0$ or $f(0,0) = 1$. If $f(0,0) = 1$ then $f(x,y) = f(x,y)f(0,0) = f(0,0) = 1$ for all $x, y \in \mathbb{R}$. The second possibility $f(0,0) = 0$ gives $f(0,y) = 0$ for all $y \in \mathbb{R}$.

(ii) If $\beta \neq 0$, we get the desired result by using Proposition 2.3 and proceeding as for (i).

Case 2: If $\beta^2 + 4\alpha \neq 0$, then we get the desired result by taking $S = (\mathbb{C}, \cdot)$ in Theorem 2.2. $\square$

As another consequence of Theorem 2.2, we express in terms of multiplicative functions on $(\mathbb{R}, \cdot)$ and additive ones on $(\mathbb{R}, +)$ the real-valued solutions of $(E(\alpha,\beta))$.

Corollary 3.2. *The general solution* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *of* $(E(\alpha,\beta))$ *depends on the sign of* $\beta^2+4\alpha$ *and is given by the following forms:*

(1) *If* $\beta^2+4\alpha=0$, *then either* $f\equiv 1$ *or there exist a multiplicative function* $M\colon (\mathbb{R},\cdot)\to\mathbb{R}$ *and an additive one* $A\colon(\mathbb{R},+)\to\mathbb{R}$ *such that for all* $x,y\in\mathbb{R}$ *we have*

(i) *For* $\beta=0$,

$$f(x,y)=\begin{cases} M(x)\exp(A(y/x)), & x\neq 0,\\ 0, & x=0.\end{cases}$$

(ii) *For* $\beta\neq 0$,

$$f(x,y)=\begin{cases} M\Big(x+\dfrac{\beta}{2}y\Big)\exp\Big(A\Big(\dfrac{\beta y}{2x+\beta y}\Big)\Big), & x+\dfrac{\beta}{2}y\neq 0,\\ 0, & \textit{else}.\end{cases}$$

(2) *If* $\beta^2+4\alpha>0$, *then for all* $(x,y)\in\mathbb{R}^2$ *we have*

$$f(x,y)=M_1\Big(x+\frac{1}{2}(\beta-\sqrt{\beta^2+4\alpha})y\Big)M_2\Big(x+\frac{1}{2}(\beta+\sqrt{\beta^2+4\alpha})y\Big),$$

where $M_1,M_2\colon(\mathbb{R},\cdot)\to\mathbb{R}$ *are multiplicative functions.*

(3) *If* $\beta^2+4\alpha<0$, *then either* $f\equiv 1$ *or there exist a multiplicative function* $M\colon(\mathbb{R}^+,\cdot)\to\mathbb{R}$ *and an additive one* $A\colon(\mathbb{R},+)\to\mathbb{R}$ *such that*

$$f(x,y)=M(x^2+\beta xy-\alpha y^2)\exp\Big(A\Big(\arctan\frac{\sqrt{-\beta^2-4\alpha}y}{2x+\beta y}\Big)\Big),$$

for all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ *and* $f(0,0)=0$.

Proof. Let $f\colon\mathbb{R}^2\to\mathbb{R}$ be a solution of $(E(\alpha,\beta))$. Depending on the sign of $\beta^2+4\alpha$, we have the following three cases:

Case 1: Suppose that $\beta^2+4\alpha=0$.

(i) If $\beta=0$ then, according to Proposition 2.3, there exist multiplicative functions $M\colon(\mathbb{R}^*,\cdot)\to\mathbb{R}$ and $\chi\colon(\mathbb{R},+)\to\mathbb{R}$ such that $f(x,y)=M(x)\chi(y/x)$ for all $(x,y)\in\mathbb{R}^*\times\mathbb{R}$. From [1, Theorem 5 in Chapter 3], the multiplicative function χ from $(\mathbb{R},+)$ to $\mathbb{R}$ has one of the following expressions

$$\chi\equiv 0 \quad\text{or}\quad \chi(x)=\exp(A(x)),\quad x\in\mathbb{R},$$

where $A\colon\mathbb{R}\to\mathbb{R}$ is an additive function. Thus $f(x,y)=M(x)\exp(A(y/x))$, $(x,y)\in\mathbb{R}^*\times\mathbb{R}$. For $f(0,y)$, $y\in\mathbb{R}$, we can proceed like in Corollary 3.1.

(ii) If $\beta \neq 0$, then we get the desired result by using Proposition 2.3 and proceeding as for (i).

Case 2: If $\beta^2+4\alpha > 0$, then we get the expected result by taking $S = (\mathbb{R}, \cdot)$ in Theorem 2.2.

Case 3: If $\beta^2 + 4\alpha < 0$ then, from Theorem 2.2, we have

$$f(x,y) = m\Big(x + \frac{1}{2}(\beta + i\sqrt{-\beta^2 - 4\alpha})y\Big), \quad x, y \in \mathbb{R}, \tag{3.1}$$

where $m\colon (\mathbb{C}, \cdot) \to \mathbb{R}$ is a multiplicative function. For all $z_1, z_2 \in \mathbb{C}$ we have

$$m(z_1 z_2) = m(z_1)m(z_2). \tag{3.2}$$

So $m(0) = 1$ or $m(0) = 0$. If $m(0) = 1$ then for $z_2 = 0$ in (3.2) we get $m \equiv 1$. Suppose now that $m(0) = 0$. Since $m(\sqrt{u_1})m(\sqrt{u_2}) = m(\sqrt{u_1 u_2})$ for all $u_1, u_2 \in \mathbb{R}^+$, then the map $M\colon (\mathbb{R}^+, \cdot) \to \mathbb{R}$ defined by

$$M(u) := m(\sqrt{u}) \quad \text{for any } u \in \mathbb{R}^+,$$

is a multiplicative function. Let $z = u + iv \in \mathbb{C}^*$ and $z = |z| \exp(i\theta)$, $\theta \in \mathbb{R}$, be its polar decomposition. We have

$$m(z) = m(|z| \exp(i\theta)) = M(|z|^2)m(\exp(i\theta)). \tag{3.3}$$

Now, for all $\theta_1, \theta_2 \in \mathbb{R}$ we have

$$m\left(\exp(i\theta_1)\right) m\left(\exp(i\theta_2)\right) = m(\exp(i(\theta_1 + \theta_2))).$$

Thus, in terms of $\psi(\theta) := m(\exp(i\theta))$, we get

$$\psi(\theta_1 + \theta_2) = \psi(\theta_1)\psi(\theta_2), \quad \theta_1, \theta_2 \in \mathbb{R}.$$

From [1, Theorem 5 in Chapter 3], ψ has one of the following two forms

$$\psi \equiv 0 \quad \text{or} \quad \psi(\theta) = \exp(A(\theta)), \quad \theta \in \mathbb{R},$$

where $A\colon \mathbb{R} \to \mathbb{R}$ is an additive function. Hence, we deduce from (3.3) that

$$\begin{aligned} m(z) &= M(|z|^2) \exp(A(\theta)) \\ &= M(|z|^2) \exp\Big(A\Big(\arctan \frac{v}{u}\Big)\Big), \end{aligned} \tag{3.4}$$

for all $z = u + iv \in \mathbb{C}^*$, where $M\colon (\mathbb{R}^+, \cdot) \to \mathbb{R}$ is a multiplicative function and $A\colon (\mathbb{R}, +) \to \mathbb{R}$ is an additive one. From (3.1) and (3.4) we conclude that, for all $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$f(x,y) = M\Big(\Big(x + \frac{1}{2}\beta y\Big)^2 - \frac{1}{4}(\beta^2 + 4\alpha)y^2\Big) \exp\Big(A\Big(\arctan \frac{\sqrt{-\beta^2 - 4\alpha}y}{2x + \beta y}\Big)\Big)$$

$$= M(x^2 + \beta xy - \alpha y^2) \exp\Big(A\Big(\arctan \frac{\sqrt{-\beta^2 - 4\alpha}y}{2x + \beta y}\Big)\Big),$$

and $f(0,0) = m(0) = 0$.

Conversely, it is elementary to prove that the formulas for f above define solutions of $(E(\alpha, \beta))$. □

- For $\mathbb{K} = \mathbb{R}$, as an immediate consequence of Corollary 3.2, taking $\beta = \lambda - 2$ and $\alpha = \lambda - 1$ where $\lambda \in \mathbb{R}^*$, we get [4, Theorem2.1].

As other interesting consequences of Corollary 3.2, on the solution $f\colon \mathbb{R}^2 \to \mathbb{R}$ of $(E(\alpha, \beta))$, we get

- [6, Theorem 1.1], [8, Corllary 3.2] and [7, Theorem 2.1] in which $(\alpha, \beta) = (-1, 1)$.
- [5, Theorem 1], here $\beta = 0$.

4. The 2×2 matrix valued solutions of $(E(\alpha, \beta))$

In this section, the range space of the solutions of $(E(\alpha, \beta))$ is the semigroup $M_2(\mathbb{C})$. The significant difference from Section 2 is that here (from Theorem 4.4, Remark 4.6, and Proposition 4.7) we can find, for $\beta^2 + 4\alpha \neq 0$, explicit expressions of the solutions $f\colon \mathbb{R}^2 \to M_2(\mathbb{C})$ of $(E(\alpha, \beta))$ in terms of scalar multiplicative functions on $\mathbb{R}$ or $\mathbb{C}$. Some numerous references concerning the study of matrix functional equations can be found e.g. in [2, 3, 10, 12, 14]. The following lemma describes the solutions of the matrix Cauchy functional equation, namely

$$M(x)M(y) = M(xy), \quad x, y \in S, \tag{4.1}$$

on a regular abelian semigroup S.

Lemma 4.1 ([10]). *Let S be a regular abelian semigroup. The solutions $M\colon S \to M_2(\mathbb{C})$ of the matrix multiplicative Cauchy functional equation* (4.1) *are the matrix valued functions of the two forms below in which P ranges over $GL_2(\mathbb{C})$:*

(1)

$$M(x) = P \begin{pmatrix} \phi_1(x) & 0 \\ 0 & \phi_2(x) \end{pmatrix} P^{-1}, \quad x \in S,$$

where $\phi_1, \phi_2 \colon S \to \mathbb{C}$ are multiplicative functions.

(2)

$$M(x) = \begin{cases} P \begin{pmatrix} \phi(x) & \phi(x)A(x) \\ 0 & \phi(x) \end{pmatrix} P^{-1} & \textit{if} \;\; x \in S \setminus I_\phi, \\ 0 & \textit{if} \;\; x \in I_\phi, \end{cases}$$

where $\phi \colon S \to \mathbb{C}$ is a multiplicative function and $A \colon S \setminus I_\phi \to \mathbb{C}$ is an additive function.

Remark 4.2. Let $\phi \colon (\mathbb{K}, \cdot) \to \mathbb{C}$ be a non-zero multiplicative function. It is easy to verify that $I_\phi = \{0\}$ or $I_\phi = \varnothing$. In fact, suppose that there exists $x_0 \neq 0$ such that $\phi(x_0) = 0$ then for all $x \in \mathbb{K} : \phi(x) = \phi(x_0)\phi(\frac{x}{x_0}) = 0$ which contradicts our assumption.

We will apply Lemma 4.1 to give the solutions $f \colon \mathbb{R}^2 \to M_2(\mathbb{C})$ of equation $(E(\alpha, \beta))$. So we will first discuss the regularity of $(\mathbb{R}^2, *_{\alpha,\beta})$.

Lemma 4.3. *Let $\alpha, \beta \in \mathbb{R}$ such that $\beta^2 + 4\alpha \neq 0$. The set $(\mathbb{R}^2, *_{\alpha,\beta})$ is a regular abelian monoid.*

Proof. Clearly $(\mathbb{R}^2, *_{\alpha,\beta})$ is an abelian monoid. In order to prove that it is regular, we will show that for all $X \in \mathbb{R}^2$ there exists $Z \in \mathbb{R}^2$ such that $X = X *_{\alpha,\beta} Z *_{\alpha,\beta} X$. Let $X = (x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} (x, y) *_{\alpha,\beta} (x + \beta y, -y) &= (x^2 + \beta xy - \alpha y^2, 0) \\ &= (x^2 + \beta xy - \alpha y^2)(1, 0). \end{aligned}$$

So we have the following two cases:

Case 1: If $x^2 + \beta xy - \alpha y^2 \neq 0$, then X is invertible and its inverse is $X^{-1} = \dfrac{1}{x^2 + \beta xy - \alpha y^2}(x + \beta y, -y)$. So it is enough to take $Z = X^{-1} \in \mathbb{R}^2$.

Case 2: Suppose that $x^2 + \beta xy - \alpha y^2 = 0$. If $y = 0$, then $X = (0, 0)$ and the result can be trivially shown. If $y \neq 0$, then we see that $\beta^2 + 4\alpha > 0$ because $\beta^2 + 4\alpha \neq 0$. Hence

$$x^2 + \beta xy - \alpha y^2 = \Big(x + \frac{1}{2}(\beta - \sqrt{\beta^2 + 4\alpha})y\Big)\Big(x + \frac{1}{2}(\beta + \sqrt{\beta^2 + 4\alpha})y\Big) = 0.$$

We first suppose that $x = -\frac{1}{2}(\beta + \sqrt{\beta^2 + 4\alpha})y$, then

$$\begin{aligned} X *_{\alpha,\beta} X &= (x^2 + \alpha y^2, 2xy + \beta y^2) \\ &= \Big(\frac{1}{4}(\beta + \sqrt{\beta^2 + 4\alpha})^2 y^2 + \alpha y^2, -\sqrt{\beta^2 + 4\alpha} y^2\Big) \\ &= -\sqrt{\beta^2 + 4\alpha} y\Big(-\frac{1}{2}(\beta + \sqrt{\beta^2 + 4\alpha})y, y\Big) \\ &= -\sqrt{\beta^2 + 4\alpha} yX. \end{aligned}$$

By using the fact that $(\mathbb{R}^2, *_{\alpha,\beta})$ is abelian and $(1, 0)$ is its neutral element, we find that

$$X = \frac{-1}{\sqrt{\beta^2 + 4\alpha}y} X *_{\alpha,\beta} X = X *_{\alpha,\beta} \big(\frac{-1}{\sqrt{\beta^2 + 4\alpha}y}(1, 0)\big) *_{\alpha,\beta} X.$$

Then, it is enough to take $Z = (\frac{-1}{\sqrt{\beta^2+4\alpha}y}, 0) \in \mathbb{R}^2$. Similarly, if $x = -\frac{1}{2}(\beta - \sqrt{\beta^2 + 4\alpha})y$, then we get that $X = X *_{\alpha,\beta} (\frac{1}{\sqrt{\beta^2 + 4\alpha}y}, 0) *_{\alpha,\beta} X$. So it is enough to take $Z = (\frac{1}{\sqrt{\beta^2+4\alpha}y}, 0)$. This completes the proof of the lemma. □

The following main theorem highlights the 2×2-matrix valued solutions of Eq. $(E(\alpha, \beta))$ for $\beta^2 + 4\alpha \neq 0$. It reads as follows:

THEOREM 4.4. *Assume $\beta^2 + 4\alpha \neq 0$. The general solution $f : \mathbb{R}^2 \to M_2(\mathbb{C})$ of $(E(\alpha, \beta))$ is given by the following expressions in which $P \in GL_2(\mathbb{C})$*

$$f(x, y) = P \begin{pmatrix} \phi_1(x, y) & 0 \\ 0 & \phi_2(x, y) \end{pmatrix} P^{-1},$$

$$f(x, y) = \begin{cases} \phi(x, y)P \begin{pmatrix} 1 & \psi(x, y) \\ 0 & 1 \end{pmatrix} P^{-1} & \text{if } (x, y) \in \mathbb{R}^2 \setminus I_\phi, \\ 0 & \text{if } (x, y) \in I_\phi, \end{cases}$$

where ϕ, ϕ_1, $\phi_2 : (\mathbb{R}^2, *_{\alpha,\beta}) \to \mathbb{C}$ are multiplicative functions and $\psi : (\mathbb{R}^2 \setminus I_\phi, *_{\alpha,\beta}) \to \mathbb{C}$ is an additive one.

PROOF. Let $f : \mathbb{R}^2 \to M_2(\mathbb{C})$ be a solution of $(E(\alpha, \beta))$ with $\beta^2 + 4\alpha \neq 0$. Then for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ we have

$$f((x_1, y_1) *_{\alpha,\beta} (x_2, y_2)) = f(x_1, y_1)f(x_2, y_2).$$

This means that, with $S = (\mathbb{R}^2, *_{\alpha,\beta})$, the function f is a solution of the matrix multiplicative Cauchy functional equation (4.1). According to Lemma 4.3 $(\mathbb{R}^2, *_{\alpha,\beta})$ is, for $\beta^2 + 4\alpha \neq 0$, a regular abelian monoid. Then the result follows immediately from Lemma 4.1. □

REMARK 4.5. Let $\phi\colon (\mathbb{R}^2, *_{\alpha,\beta}) \to \mathbb{C}$ be a non-zero multiplicative function. It is easy to verify, by using Corollary 3.1 and Remark 4.2, that

(1) If $\beta^2 + 4\alpha < 0$, then either $I_\phi = \varnothing$ (in this case $\phi \equiv 1$) or $I_\phi = \{(0,0)\}$.
(2) If $\beta^2 + 4\alpha > 0$, then either $I_\phi = \varnothing$ or

$$I_\phi = \Big\{\Big(-\frac{1}{2}(\beta \mp \sqrt{\beta^2 + 4\alpha})y, y\Big) \mid y \in \mathbb{R}\Big\}.$$

REMARK 4.6. The multiplicative functions $\phi\colon (\mathbb{R}^2, *_{\alpha,\beta}) \to \mathbb{C}$ (i.e. the solutions $\phi\colon \mathbb{R}^2 \to \mathbb{C}$ of $(E(\alpha,\beta))$) are given in Corollary 3.1 (2) and (3). Then, from Theorem 4.4, in order to get the explicit expressions of the solutions $f\colon (\mathbb{R}^2, *_{\alpha,\beta}) \to M_2(\mathbb{C})$ of $(E(\alpha,\beta))$ it remains to determine, for a fixed multiplicative function $\phi\colon (\mathbb{R}^2, *_{\alpha,\beta}) \to \mathbb{C}$, the solution $\psi\colon \mathbb{R}^2 \setminus I_\phi \to \mathbb{C}$ of the Cauchy's additive $*_{\alpha,\beta}$-functional equation

$$\psi((x_1,y_1) *_{\alpha,\beta} (x_2,y_2)) = \psi(x_1,y_1) + \psi(x_2,y_2). \tag{4.2}$$

Clearly, if $I_\phi = \varnothing$ then $\psi \equiv 0$ because $\psi(x,y) + \psi(0,0) = \psi(0,0)$ for all $(x,y) \in \mathbb{R}^2$. So in the following proposition we work with $I_\phi \neq \varnothing$.

PROPOSITION 4.7. *Assume that $\beta^2 + 4\alpha \neq 0$ and let $\phi\colon (\mathbb{R}^2, *_{\alpha,\beta}) \to \mathbb{C}$ be a fixed non-zero multiplicative function such that $I_\phi \neq \varnothing$. The general solution $\psi\colon \mathbb{R}^2 \setminus I_\phi \to \mathbb{C}$ of (4.2) depends on the sign of $\beta^2 + 4\alpha$ and is given by:*

(1) *If $\beta^2 + 4\alpha < 0$, then there exists an additive function $A\colon (\mathbb{C}^*, \cdot) \to \mathbb{C}$ such that*

$$\psi(x,y) = A\Big(x + \frac{1}{2}(\beta + i\sqrt{-\beta^2 - 4\alpha})y\Big),$$

for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(2) *If $\beta^2 + 4\alpha > 0$, then there exist additive functions $A_1, A_2\colon (\mathbb{R}^*, \cdot) \to \mathbb{C}$ such that*

$$\psi(x,y) = A_1\Big(x + \frac{\beta - \sqrt{\beta^2 + 4\alpha}}{2}y\Big) + A_2\Big(x + \frac{\beta + \sqrt{\beta^2 + 4\alpha}}{2}y\Big),$$

for all $(x,y) \in \mathbb{R}^2 \setminus I_\phi$ and here

$$I_\phi = \Big\{\Big(-\frac{1}{2}(\beta \mp \sqrt{\beta^2 + 4\alpha})y, y\Big) \mid y \in \mathbb{R}\Big\}.$$

Proof. Let $\psi\colon \mathbb{R}^2 \setminus I_\phi \to \mathbb{C}$ be a solution of equation (4.2) such that $\beta^2 + 4\alpha \neq 0$. In what follows we distinguish between two cases:

Case 1: Suppose that $\beta^2 + 4\alpha < 0$. From Remark 4.5 (1) we have $I_\phi = \{(0,0)\}$ because here $I_\phi \neq \varnothing$. For all $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we define the function $\Phi\colon \mathbb{H}^* \to \mathbb{C}$ by

$$\Phi(a+ib, a-ib) := \psi\Big(a - \frac{\beta}{\sqrt{-\beta^2-4\alpha}}b, \frac{2}{\sqrt{-\beta^2-4\alpha}}b\Big),$$

where $\mathbb{H}^* := \{(z,\bar{z}) \mid z \in \mathbb{C}^*\}$, this is equivalent to

$$\psi(x,y) = \Phi\Big(x + \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})y, x + \frac{1}{2}(\beta - i\sqrt{-\beta^2-4\alpha})y\Big), \tag{4.3}$$

where $x, y \in \mathbb{R}$. For any $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we compute that

$$\begin{aligned}
&\Phi\Big(x_1 + \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})y_1, x_1 + \frac{1}{2}(\beta - i\sqrt{-\beta^2-4\alpha})y_1\Big)\\
&+ \Phi\Big(x_2 + \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})y_2, x_2 + \frac{1}{2}(\beta - i\sqrt{-\beta^2-4\alpha})y_2\Big)\\
&= \psi(x_1,y_1) + \psi(x_2,y_2)\\
&= \psi(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1 + \beta y_1y_2)\\
&= \Phi\bigg(\Big(x_1 + \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})y_1\Big)\Big(x_2 + \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})y_2\Big),\\
&\qquad \Big(x_1 + \frac{1}{2}(\beta - i\sqrt{-\beta^2-4\alpha})y_1\Big)\Big(x_2 + \frac{1}{2}(\beta - i\sqrt{-\beta^2-4\alpha})y_2\Big)\bigg).
\end{aligned}$$

This means that, for all $z_1, z_2 \in \mathbb{C}^*$, we have

$$\Phi(z_1,\bar{z}_1) + \Phi(z_2,\bar{z}_2) = \Phi(z_1z_2, \overline{z_1z_2}),$$

which yields that the function $A\colon (\mathbb{C}^*,\cdot) \to \mathbb{C}$ defined by

$$A(z) = \Phi(z,\bar{z}) \quad \text{for all } z \in \mathbb{C}^*, \tag{4.4}$$

is additive. Therefore, from (4.4) and (4.3), we infer that

$$\psi(x,y) = A\Big(x + \frac{1}{2}(\beta + i\sqrt{-\beta^2-4\alpha})y\Big), \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Case 2: Suppose that $\beta^2 + 4\alpha > 0$. Define $\sigma\colon \mathbb{R}^2 \setminus I_\phi \to \mathbb{R}^* \times \mathbb{R}^*$ by

$$\sigma(x,y) := \Big(x + \frac{1}{2}(\beta - \sqrt{\beta^2+4\alpha})y, x + \frac{1}{2}(\beta + \sqrt{\beta^2+4\alpha})y\Big),$$

for all $(x,y) \in \mathbb{R}^2 \setminus I_\phi$, and let $\odot$ be the binary operation on $\mathbb{R}^* \times \mathbb{R}^*$ defined by

$$(x_1,y_1) \odot (x_2,y_2) = (x_1x_2, y_1y_2), \quad (x_1,y_1),(x_2,y_2) \in \mathbb{R}^* \times \mathbb{R}^*.$$

According to Remark 4.5 (2) and the fact that $I_\phi \neq \varnothing$, we can easily prove that σ is a bijective homomorphism from $(\mathbb{R}^2 \setminus I_\phi, *_{\alpha,\beta})$ to $(\mathbb{R}^* \times \mathbb{R}^*, \odot)$. Let $\Psi\colon \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{C}$ be a function defined by

$$\Psi := \psi \circ \sigma^{-1}. \tag{4.5}$$

From (4.5) we reformulate (4.2) in terms of Ψ as

$$\Psi \circ \sigma((x_1,y_1) *_{\alpha,\beta} (x_2,y_2)) = \Psi \circ \sigma(x_1,y_1) + \Psi \circ \sigma(x_2,y_2),$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2 \setminus I_\phi$. This yields that

$$\Psi(\sigma(x_1,y_1) \odot \sigma(x_2,y_2)) = \Psi(\sigma(x_1,y_1)) + \Psi(\sigma(x_2,y_2)).$$

Hence, for all $(u_1,v_1),(u_2,v_2) \in \mathbb{R}^* \times \mathbb{R}^*$ we have

$$\Psi(u_1,v_1) + \Psi(u_2,v_2) = \Psi((u_1,v_1) \odot (u_2,v_2)) = \Psi(u_1u_2, v_1v_2).$$

By using the last equality, we conclude that the functions $x \mapsto \Psi(x,1)$ and $y \mapsto \Psi(1,y)$ are additive functions from $(\mathbb{R}^*,\cdot)$ to $(\mathbb{R},+)$ and that

$$\Psi(x,y) = \Psi(x,1) + \Psi(1,y), \quad x,y \in \mathbb{R}^*.$$

So there exist additive functions $A_1, A_2\colon (\mathbb{R}^*,\cdot) \to (\mathbb{R},+)$ such that

$$\Psi(x,y) = A_1(x) + A_2(y), \quad x,y \in \mathbb{R}^*. \tag{4.6}$$

Therefore, from (4.5) and (4.6), we conclude that

$$\begin{aligned}\psi(x,y) &= \Psi \circ \sigma(x,y)\\ &= \Psi\Big(x + \frac{1}{2}(\beta - \sqrt{\beta^2+4\alpha})y, x + \frac{1}{2}(\beta + \sqrt{\beta^2+4\alpha})y\Big)\\ &= A_1\Big(x + \frac{1}{2}(\beta - \sqrt{\beta^2+4\alpha})y\Big) + A_2\Big(x + \frac{1}{2}(\beta + \sqrt{\beta^2+4\alpha})y\Big).\end{aligned}$$

The converse statement is straightforward. □

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References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and its Applications, 31, Cambridge University Press, Cambridge, 1989.

[2] Y. Aissi and D. Zeglami, *D'Alembert's matrix functional equation with an endomorphism on abelian groups*, Results Math. **75** (2020), no. 4, Paper No. 137, 17 pp.

[3] Y. Aissi, D. Zeglami, and M. Ayoubi, *A variant of d'Alemberts matrix functional equation*, Ann. Math. Sil. **35** (2021), no. 1, 21–43.

[4] M. Akkouchi and M.H. Lalaoui Rhali, *General solutions of some functional equations*, Bol. Mat. **12** (2005), no 1, 57–62.

[5] L.R. Berrone and L.V. Dieulefait, *A functional equation related to the product in a quadratic number field*, Aequationes Math. **81** (2011), no. 1–2, 167–175.

[6] E.A. Chávez and P.K. Sahoo, *On functional equations of the multiplicative type*, J. Math. Anal. Appl. **431** (2015), no. 1, 283–299.

[7] J. Chung and P.K. Sahoo, *On a functional equation arising from Proth identity*, Commun. Korean Math. Soc. **31** (2016), no. 1, 131–138.

[8] B. Ebanks, *Reconsidering a functional equation arising from number theory*, Publ. Math. Debrecen, **86** (2015), no. 1-2, 135–147.

[9] S.-M. Jung and J.-H. Bae, *Some functional equations originating from number theory*, Proc. Indian Acad. Sci. Math. Sci. **113** (2003), no. 2, 91–98.

[10] A. Mouzoun, D. Zeglami, and Y. Aissi, *Matrix homomorphism equations on a class of monoids and non Abelian groupoids*, J. Math. Anal. Appl. **503** (2021), no. 2, Paper No. 125354, 24 pp.

[11] A. Mouzoun, D. Zeglami, and M. Ayoubi, *A functional equation originated from the product in a cubic number field*, Mediterr. J. Math. **18** (2021), no. 5, Paper No. 191, 14 pp.

[12] P. Sinopoulos, *Wilson's functional equation for vector and matrix functions*, Proc. Amer. Math. Soc. **125** (1997), no. 4, 1089–1094.

[13] H. Stetkær, *Functional Equations on Groups*, World Scientific Publishing Company, Singapore, 2013.

[14] H. Stetkær, *D'Alembert's and Wilson's functional equations for vector and 2×2 matrix valued functions*, Math. Scand. **87** (2000), no. 1, 115–132.

[15] D. Zeglami, *Some functional equations related to number theory*, Acta Math. Hungar. **149** (2016), no. 2, 490–508.

Department of Mathematics
E.N.S.A.M
Moulay ISMAÏL University
B.P: 15290 Al Mansour
Meknès
Morocco
e-mail: mouzounposte@gmail.com
e-mail: zeglamidriss@yahoo.fr
e-mail: yaissi94@gmail.com