Annales Mathematicae Silesianae **35** (2021), no. 1, 44–54
DOI: 10.2478/amsil-2020-0027

# A NOTE ON THE ASYMPTOTIC BEHAVIOR OF THE DISTRIBUTION FUNCTION OF A GENERAL SEQUENCE

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**Abstract.** The aim of this note is to study the distribution function of certain sequences of positive integers, including, for example, Bell numbers, factorials and primorials. In fact, we establish some general asymptotic formulas in this regard. We also prove some limits that connect these sequences with the number $e$. Furthermore, we present a generalization of the primorial.

## 1. Introduction

In combinatorics, the Bell number $B_n$ counts the number of different ways to partition a set with $n$ elements, where $n \in \mathbb{N} \cup \{0\}$. Bell numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell (1883–1960), Scottish mathematician, who wrote some comprehensive papers about them in the 1930s (see [4] and [3]). The Bell numbers satisfy the following recurrence relation [2, page 216]:

$$B_0 = 1, \quad B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} B_i, \quad \forall n \in \mathbb{N}.$$

*Received: 14.06.2020. Accepted: 29.12.2020. Published online: 26.01.2021.*
(2020) Mathematics Subject Classification: 97I30, 11B83, 11B73, 11B99, 11B65, 11A99, 40A05.
*Key words and phrases:* Bell numbers, factorials, primorials, distribution function, asymptotic behavior, the number $e$.

The first few Bell numbers are $B_0 = 1$, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$, $B_6 = 203$, $B_7 = 877$, $B_8 = 4140$, $B_9 = 21147$, $B_{10} = 115975$ (see sequence A000110 in the OEIS).

As $n$ grows, it is gradually becoming harder to calculate the Bell numbers. Hence, knowing how Bell numbers are distributed among the integers when $n \to \infty$ helps us to be aware of the growth rate of these numbers. Let $\omega(x)$ be the number of $B_n$ not exceeding $x$, that is $\omega(x)$ is the distribution or counting function of Bell numbers. Clearly, studying the function $\omega(x)$ as $x \to \infty$ helps us to understand how Bell numbers are distributed among integers. A first approach is given by Jakimczuk in [9]. He proved that

$$\omega(x) \sim \frac{\ln x}{\ln \ln x} \quad (x \to \infty), \tag{1.1}$$

i.e.,

$$\lim_{x\to\infty} \frac{\omega(x)}{\frac{\ln x}{\ln \ln x}} = 1.$$

In fact, Jakimczuk more generally showed that if a strictly increasing sequence of integers such as $F_n$ satisfy the asymptotic formula

$$\log F_n \sim cn \log n \quad (c > 0),$$

then the number of $F_n$ that do not exceed $n$ is asymptotically equivalent to $\frac{\log n}{c \log \log n}$ as $n \to \infty$. Thus, since $\log B_n \sim n \log n$, then equation (1.1) holds.

In this paper, we establish some new generalizations and we also show that the relation (1.1) is true for some other known sequences as the sequence of factorials and the sequence of primorials (see sequences A000142 and A002110 in the OEIS). We also study the asymptotic behavior of the sum of the distribution functions of these sequences. Furthermore, a generalization of the primorial will be established.

## 2. Main results

In this section we aim to present our main results. First, we shall prove a general theorem on the distribution function of certain sequences of fast increase.

Theorem 2.1. *Let $A_n$ be a strictly increasing sequence of positive integers such that*

$$\log A_n \sim cn^k \log n, \quad c > 0,\ k \in \mathbb{N}. \tag{2.1}$$

*If $\varphi(x)$ is the distribution function of the sequence $A_n$ (i.e., $\varphi(x)$ is the number of $A_n$ not exceeding $x$), then the following asymptotic formula holds:*

$$\varphi(x) \sim \left(\frac{k}{c}\frac{\log x}{\log\log x}\right)^{\frac{1}{k}}. \tag{2.2}$$

Proof. By (2.1), we have

$$\log\log A_n \sim k \log n. \tag{2.3}$$

Equations (2.1) and (2.3) give

$$\left(\frac{k}{c}\frac{\log A_n}{\log\log A_n}\right)^{\frac{1}{k}} \sim n = \varphi(A_n). \tag{2.4}$$

Note that by equations (2.1) and (2.3), we have

$$\log A_{n+1} \sim \log A_n, \quad \log\log A_{n+1} \sim \log\log A_n. \tag{2.5}$$

Now, suppose that $x \in [A_n, A_{n+1})$. Then equations (2.4) and (2.5) give

$$1 \leftarrow \frac{\varphi(A_n)}{\left(\frac{k}{c}\frac{\log A_{n+1}}{\log\log A_{n+1}}\right)^{\frac{1}{k}}} \leq \frac{\varphi(x)}{\left(\frac{k}{c}\frac{\log x}{\log\log x}\right)^{\frac{1}{k}}} \leq \frac{\varphi(A_n)}{\left(\frac{k}{c}\frac{\log A_n}{\log\log A_n}\right)^{\frac{1}{k}}} \to 1.$$

Equation (2.2) is proved. Note that the function $\left(\frac{k}{c}\frac{\log x}{\log\log x}\right)^{\frac{1}{k}}$ (with $c > 0$, $k \in \mathbb{N}$) is strictly increasing from a certain value of $x$. On the other hand, note that if $x \in [A_n, A_{n+1})$, then $\varphi(x) = \varphi(A_n)$. The theorem is proved. □

There are many sequences of positive integers that satisfy Theorem 2.1. The more simple are $A_n = n^{(n^k)}$, where $k \in \mathbb{N}$. In particular, if $k = 1$ we obtain $A_n = n^n$. Another example is the sequence of Bell numbers which holds under Theorem 2.1 with $c = 1$ and $k = 1$ (see [9]). In the following theorem we prove that two other well-known sequences of interest in number theory satisfy Theorem 2.1.

THEOREM 2.2. *Let us consider the sequences $A_n = n!$ (sequence of factorials), and $A_n = P_n = p_1 p_2 \cdots p_n$, where $p_n$ is the nth prime number (sequence of primorials). Then, the distribution functions $\varphi(x)$ of these sequences satisfy*

$$\varphi(x) \sim \frac{\log x}{\log\log x}. \tag{2.6}$$

PROOF. The Stirling's formula $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$ (see [1]) has the weak consequence

$$\log n! = \sum_{i=1}^{n} \log i = n \log n - n + o(n). \tag{2.7}$$

Therefore, by Theorem 2.1 the distribution function of the sequence of factorials $A_n = n!$ satisfies equation (2.6).

Now, we shall consider the sequence $P_n$ of primorials. It is well-known that (see [7])

$$\log P_n = \sum_{i=1}^{n} \log p_i = n \log n + n \log\log n - n + o(n). \tag{2.8}$$

Therefore by Theorem 2.1 the distribution function of the sequence of primorials $A_n = P_n$ satisfies equation (2.6). The theorem is proved. □

Note that the function in equation (2.2) and consequently its particular case in equation (2.6) are functions of slow increase (for more details about the functions of slow increase, we refer the reader to [8]). Based on this property we prove the following theorem.

THEOREM 2.3. *Suppose that the distribution function $\varphi(x)$ of a sequence satisfies equation* (2.2) *or in particular equation* (2.6). *Then the following asymptotic formula holds:*

$$\sum_{i=1}^{n} \varphi(i) \sim n \left( \frac{k}{c} \frac{\log n}{\log\log n} \right)^{\frac{1}{k}}. \tag{2.9}$$

*In particular, if $c = k = 1$, we have*

$$\sum_{i=1}^{n} \varphi(i) \sim \frac{n \log n}{\log\log n}. \tag{2.10}$$

Proof. First, let us recall the well-known proposition (see [10, page 332]) that states for two series of positive terms $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$, if $\sum_{i=1}^{\infty} b_i$ diverges and $a_i \sim b_i$, then $\sum_{i=1}^{n} a_i \sim \sum_{i=1}^{n} b_i$. Now, using this fact and by use of (2.2), we have

$$\sum_{i=1}^{n} \varphi(i) \sim \sum_{i=h}^{n} \left( \frac{k}{c} \frac{\log i}{\log \log i} \right)^{\frac{1}{k}} . \tag{2.11}$$

Since $\left( \frac{k}{c} \frac{\log x}{\log \log x} \right)^{\frac{1}{k}}$ (with $c > 0$ and $k \in \mathbb{N}$) is strictly increasing and positive in the interval $[h, \infty)$ (with $h > e \approx 2.71828 \ldots$), we find that

$$\sum_{i=h}^{n} \left( \frac{k}{c} \frac{\log i}{\log \log i} \right)^{\frac{1}{k}} = \int_{h}^{n} \left( \frac{k}{c} \frac{\log x}{\log \log x} \right)^{\frac{1}{k}} dx + O\left( \left( \frac{k}{c} \frac{\log n}{\log \log n} \right)^{\frac{1}{k}} \right), \tag{2.12}$$

and since $\left( \frac{k}{c} \frac{\log x}{\log \log x} \right)^{\frac{1}{k}}$ is a function of slow increase, therefore by [8, Theorem 7] we have

$$\int_{h}^{x} \left( \frac{k}{c} \frac{\log t}{\log \log t} \right)^{\frac{1}{k}} dt \sim x \left( \frac{k}{c} \frac{\log x}{\log \log x} \right)^{\frac{1}{k}}$$

and consequently

$$\sum_{i=h}^{n} \left( \frac{k}{c} \frac{\log i}{\log \log i} \right)^{\frac{1}{k}} \sim n \left( \frac{k}{c} \frac{\log n}{\log \log n} \right)^{\frac{1}{k}} . \tag{2.13}$$

Hence, (2.13) and (2.11) give (2.9). The theorem is proved. □

Remark 2.4. Equation (2.10) holds, for example, for the sequence of Bell numbers, factorials and primorials (see Theorem 2.2).

According to the proof of Theorem 2.3 (case $k = c = 1$), since $\varphi(x) \sim \frac{\log x}{\log \log x}$, therefore $\sum_{i=1}^{n} \varphi(i) \sim \sum_{i=h}^{n} \frac{\log i}{\log \log i}$. Hence, studying the asymptotic behavior of $\sum_{i=h}^{n} \frac{\ln i}{\ln \ln i}$ will be useful for studying the asymptotic behavior of $\sum_{i=1}^{n} \varphi(i)$. In the next theorem we obtain an asymptotic expansion for the sum $\sum_{i=h}^{n} \frac{\log i}{\log \log i}$.

THEOREM 2.5. *For each positive integer $N$, we have*

$$(2.14)\quad \sum_{i=h}^{n} \frac{\log i}{\log\log i} = \frac{n\log n}{\log\log n} - n\frac{\log\log n - 1}{(\log\log n)^2} - n\sum_{k=1}^{N} \frac{T_k(\log\log n)}{(\log n)^k(\log\log n)^{k+2}} + O\left(\frac{n}{(\log n)^{N+1}(\log\log n)^2}\right),$$

*where $T_k(x)$ $(k \geq 1)$ is a polynomial of degree $k$ and leading coefficient $(k-1)!$. These polynomials can be obtained by the recurrence formula*

$$T_{m+1}(x) = \begin{cases} x-2, & if\ m=0; \\ T_m(x)(mx+(m+2)) - xT'_m(x), & if\ m \geq 1. \end{cases}$$

*Thus, $T_1 = x-2$, $T_2(x) = x^2-6$, $T_3(x) = 2x^3+2x^2-12x-24$, $T_4(x) = 6x^4+10x^3-30x^2-120x-120$, etc.*

PROOF. By (2.12) (with $k=c=1$), we have

$$(2.15)\quad \sum_{i=h}^{n} \frac{\log i}{\log\log i} = \int_h^n \frac{\log x}{\log\log x}\,dx + O\left(\frac{\log n}{\log\log n}\right).$$

By integration by parts twice, we obtain

$$(2.16)\quad \int \frac{\log x}{\log\log x}\,dx = \frac{x\log x}{\log\log x} - x\frac{\log\log x - 1}{(\log\log x)^2} - \int \frac{T_1(\log\log x)}{\log x(\log\log x)^3}\,dx.$$

If $m \geq 1$, then by integration by parts we obtain

$$(2.17)\quad \int \frac{T_m(\log\log x)}{(\log x)^m(\log\log x)^{m+2}}\,dx = \frac{xT_m(\log\log x)}{(\log x)^m(\log\log x)^{m+2}} + \int \frac{T_{m+1}(\log\log x)}{(\log x)^{m+1}(\log\log x)^{m+3}}\,dx.$$

By successive application of (2.17) into (2.16) and by use of (2.15) we obtain (2.14), since

$$\int_h^n \frac{T_{N+1}(\log\log x)}{(\log x)^{N+1}(\log\log x)^{N+3}}\,dx = O\left(\frac{n}{(\log n)^{N+1}(\log\log n)^2}\right).$$

Note that any function of the form $f(x) = (\log x)^s(\log\log x)^m$, where $s$ and $m$ are positive integers, is of slow increase, and by [8, Equation (15)], for any slow increasing function $f(x)$ we have $\int_a^x \frac{1}{f(t)}\,dt \sim \frac{x}{f(x)}$. The theorem is proved. □

Next, we prove some limits that connect the sequences of factorials, primorials, and Bell numbers to the number $e(\approx 2.71828\ldots)$.

THEOREM 2.6. *Let $p_n$ denote the nth prime number and $\pi(.)$ denote the distribution function of prime numbers. If $n!$, $P_n$, and $B_n$ are the nth factorial, nth primorial, and nth Bell number, respectively, then the following limits hold:*

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^n i!\right)^{\frac{2}{n^2}}}{n} = \frac{1}{\sqrt{e^3}}, \tag{2.18}$$

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^n P_i\right)^{\frac{2}{n^2}}}{n\log n} = \frac{1}{\sqrt{e^3}}, \tag{2.19}$$

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^n P_i\right)^{\frac{2}{n^2}}}{p_n} = \frac{1}{\sqrt{e^3}}, \tag{2.20}$$

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^n B_i\right)^{\frac{2}{n^2}}}{\frac{n}{\log n}} = \frac{1}{\sqrt{e^3}}, \tag{2.21}$$

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^n B_i\right)^{\frac{2}{n^2}}}{\pi(n)} = \frac{1}{\sqrt{e^3}}. \tag{2.22}$$

PROOF. Equation (2.22) is an immediate consequence of equation (2.21) and the well-known prime number theorem $\pi(n) \sim \frac{n}{\log n}$. Equation (2.20) is an immediate consequence of equation (2.19) and the well-known prime number theorem $p_n \sim n\log n$. We shall prove equation (2.21) by use of the following well-known equation (see [5])

$$\log B_n = n\log n - n\log\log n - n + o(n). \tag{2.23}$$

The proofs of equations (2.18) and (2.19) are the same by use of equations (2.7) and (2.8). Using (2.23) we have

$$\begin{aligned} \log\left(\prod_{i=1}^n B_i\right) &= \sum_{i=1}^n \log B_i \\ &= \sum_{i=1}^n i\log i - \sum_{i=1}^n i\log\log i - \sum_{i=1}^n i + \sum_{i=1}^n o(i). \end{aligned} \tag{2.24}$$

Since the function $x \log x$ is strictly increasing and positive in the interval $[1, \infty)$, we find that

$$\sum_{i=1}^{n} i \log i = \int_1^n x \log x \ dx + O(n \log n) = \frac{n^2}{2} \log n - \frac{n^2}{4} + o(n^2). \tag{2.25}$$

A similar argument shows that

$$\begin{aligned} \sum_{i=1}^{n} i \log\log i &= \int_1^n x \log\log x \ dx + O(n \log\log n) \\ &= \frac{n^2}{2} \log\log n + o(n^2). \end{aligned} \tag{2.26}$$

On the other hand, we have (see, for example, [6, Equation (2.7)])

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + o(n^2) \tag{2.27}$$

and we have also

$$\sum_{i=1}^{n} o(i) = o\left(\sum_{i=1}^{n} i\right) = o(n^2). \tag{2.28}$$

Hence, (2.24)–(2.28) give

$$\begin{aligned} &\log\left(\frac{(\prod_{i=1}^{n} B_i)^{\frac{2}{n^2}}}{\frac{n}{\log n}}\right) \\ &= \frac{2}{n^2}\left(\sum_{i=1}^{n} i \log i - \sum_{i=1}^{n} i \log\log i - \sum_{i=1}^{n} i + \sum_{i=1}^{n} o(i)\right) - \log(\frac{n}{\log n}) \\ &= \frac{2}{n^2}\left(\frac{n^2 \log n}{2} - \frac{n^2}{4} - \frac{n^2 \log\log n}{2} - \frac{n^2}{2} + o(n^2)\right) - \log(\frac{n}{\log n}) \\ &= -\frac{3}{2} + o(1), \end{aligned}$$

which implies (2.21). The theorem is proved. □

We know that the $n$th primorial number is defined by $P_n = p_1p_2\cdots p_n$, where $p_n$ denotes the $n$th prime number. On the other hand, by the well-known prime number theorem we have $p_n \sim n\log n$, and we also know that $\log n$ is of slow increase (see [8]). Here, we present a generalization of the primorial and prove that this generalization satisfies equation (2.2).

Theorem 2.7 (Generalization of the primorial). *Let $q_n$ is a strictly increasing sequence of positive integers such that*

$$q_n \sim n^s f(n) \quad (s \geq 1), \tag{2.29}$$

*where $f(x)$ is a function of slow increase.*

*Now, if $Q_n = q_1q_2\cdots q_n$, and $\varphi(x)$ is the distribution function of the sequence $Q_n$, then*

$$\varphi(x) \sim \frac{1}{s}\frac{\log x}{\log\log x}. \tag{2.30}$$

Proof. Following [8, Theorem 24], we have

$$\log Q_n = \log\left(\prod_{i=1}^{n} q_i\right) = \sum_{i=1}^{n}\log q_i = sn\log n + n\log f(n) - sn + o(n). \tag{2.31}$$

Therefore, equation (2.30) is obtained by (2.31) and (2.2) (with $k = 1$ and $c = s$). The theorem is proved. □

Thus, the known primorial is a special case of Theorem 2.7 when $s = 1$ and $f(n) = \log n$.

Here, we provide generalizations of equations (2.19) and (2.20) for the sequence defined in Theorem 2.7.

Theorem 2.8. *Let $Q_n$ be the same sequence defined in Theorem 2.7. Then the following limits hold:*

$$\lim_{n\to\infty}\frac{\left(\prod_{i=1}^{n} Q_i\right)^{\frac{2}{n^2}}}{n^s f(n)} = \frac{1}{\sqrt{e^{3s}}}, \tag{2.32}$$

$$\lim_{n\to\infty}\frac{\left(\prod_{i=1}^{n} Q_i\right)^{\frac{2}{n^2}}}{q_n} = \frac{1}{\sqrt{e^{3s}}}. \tag{2.33}$$

PROOF. Equation (2.33) is an immediate consequence of equations (2.32) and (2.29). Therefore, we shall prove equation (2.32). We have (see equation (2.31))

$$\begin{aligned}\log\left(\prod_{i=1}^{n} Q_i\right) &= \sum_{i=1}^{n}\log Q_i \\ &= s\sum_{i=1}^{n} i\log i + \sum_{i=1}^{n} i\log f(i) - s\sum_{i=1}^{n} i + \sum_{i=1}^{n} o(i).\end{aligned} \tag{2.34}$$

We have also (see (2.25))

$$\sum_{i=1}^{n} i\log i = \int_1^n x\log x\ dx + O(n\log n) = \frac{n^2}{2}\log n - \frac{n^2}{4} + o(n^2). \tag{2.35}$$

Since $f(x)$ is of slow increase (by Theorem 2.7), hence the function $xf(x)$ is increasing and therefore we have (integration by parts)

$$\begin{aligned}\sum_{i=1}^{n} i\log f(i) &= \int_1^n x\log f(x)\ dx + O(n\log f(n)) \\ &= \frac{n^2}{2}\log f(n) + o(n^2).\end{aligned} \tag{2.36}$$

On the other hand, we have $\sum_{i=1}^{n} i = \frac{n^2}{2} + o(n^2)$ and $\sum_{i=1}^{n} o(i) = o\left(\sum_{i=1}^{n} i\right)$ $= o(n^2)$ (see (2.27) and (2.28)). Hence, (2.34)–(2.36), (2.27), and (2.28) give

$$\begin{aligned}&\log\left(\frac{\left(\prod_{i=1}^{n} Q_i\right)^{\frac{2}{n^2}}}{n^s f(n)}\right) \\ &= \frac{2}{n^2}\left(s\sum_{i=1}^{n} i\log i + \sum_{i=1}^{n} i\log f(i) - s\sum_{i=1}^{n} i + \sum_{i=1}^{n} o(i)\right) - \log(n^s f(n)) \\ &= \frac{2}{n^2}\left(\frac{sn^2\log n}{2} - \frac{sn^2}{4} + \frac{n^2\log f(n)}{2} - \frac{sn^2}{2} + o(n^2)\right) - \log(n^s f(n)) \\ &= -\frac{3}{2}s + o(1)\end{aligned}$$

which implies (2.32). The theorem is proved. □

**Acknowledgment.** The authors would like to thank the editor and the anonymous referee for their valuable comments.

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