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A FURTHER GENERALIZATION OF $\lim_{n\to\infty} \sqrt[n]{n!}/n = 1/e$

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Abstract. It is well-known, as follows from the Stirling's approximation $n! \sim \sqrt{2\pi n}(n/e)^n$, that $\sqrt[n]{n!}/n \to 1/e$. A generalization of this limit is $(1^{1^s} \cdot 2^{2^s} \cdots n^{n^s})^{1/n^{s+1}} \cdot n^{-1/(s+1)} \to e^{-1/(s+1)^2}$ which was established by N. Schaumberger in 1989 (see [8]). The aim of this work is to establish a new generalization that is in fact an improvement of Schaumberger's formula for a general sequence A_n of positive real numbers. All of the results are applied to some well-known sequences in mathematics, for example, for the prime numbers sequence and the sequence of perfect powers.

1. Introduction

One of the well-known consequences of the Stirling's approximation $n! \sim \sqrt{2\pi n}(n/e)^n$ is the following limit formula (see, e.g., [3], [5] and [7]):

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}. \tag{1.1}$$

In 1989, N. Schaumberger ([8]) established a generalized form of (1.1). He proved that for any $s \in \mathbb{N} \cup \{0\}$,

$$\lim_{n\to\infty} \frac{\left(1^{1^s} \cdot 2^{2^s} \cdots n^{n^s}\right)^{1/n^{s+1}}}{n^{1/(s+1)}} = e^{-1/(s+1)^2}. \tag{1.2}$$

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Obviously, if one puts $s = 0$ in limit formula (1.2), then limit formula (1.1) is obtained.

In this note, we aim to generalize the Schaumberger's formula for a general sequence A_n of positive real numbers. Thus, we will show that our generalization gives limit formulas (1.1) and (1.2) when $A_n = n$. Furthermore, we show that the new generalization applies to some well-known sequences in mathematics, for example, for the prime numbers sequence and the sequence of perfect powers.

2. Main results

In this section we aim to present our main results. First, let us consider the following lemma:

LEMMA 2.1 ([6, page 332]). *Let* $\sum_{i=1}^{\infty} a_i$ *and* $\sum_{i=1}^{\infty} b_i$ *be two series of positive terms such that* $\frac{a_i}{b_i} \to 0$ *and* $\sum_{i=1}^{\infty} b_i$ *is divergent. Then* $\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \to 0$.

THEOREM 2.2. *Let* A_n *be a strictly increasing sequence of positive real numbers tending to infinity satisfying the asymptotic formula* $A_n \sim A_{n+1}$ *(i.e.,* $\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = 1$*), and let* $d_n = A_{n+1} - A_n$*. Then for any integer* $s > 0$,

$$\frac{\left(\prod_{i=1}^{n-1} A_i^{d_i A_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}} \to e^{-1/(s+1)^2} \tag{2.1}$$

if and only if

$$\frac{\sum_{i=1}^{n-1} A_i^{s-1}\ d_i^2 \log A_i}{A_n^{s+1}} \to 0. \tag{2.2}$$

Hint: note that condition $A_n \sim A_{n+1}$ is equivalent to the condition $\frac{d_n}{A_n} \to 0$.

PROOF. We have the following formula:

$$\int x^s \log x \, dx = \frac{x^{s+1}}{s+1} \log x - \frac{x^{s+1}}{(s+1)^2} + c \tag{2.3}$$

and the following two formulas (use L'Hospital's rule)

$$\log(1+x) = x - \frac{1}{2}x^2 + o\left(x^2\right) \quad (x \to 0), \tag{2.4}$$

$$(1+x)^{s+1} = 1 + (s+1)x + \frac{s(s+1)}{2}x^2 + o\left(x^2\right) \quad (x \to 0). \tag{2.5}$$

Hence, (2.3), (2.4), and (2.5) give

$$\begin{aligned}
\int_{A_i}^{A_{i+1}} x^s \log x \, dx &= \frac{A_{i+1}^{s+1}}{s+1} \log A_{i+1} - \frac{A_{i+1}^{s+1}}{(s+1)^2} - \frac{A_i^{s+1}}{s+1} \log A_i + \frac{A_i^{s+1}}{(s+1)^2} \\
&= \frac{A_i^{s+1}}{s+1} \log A_i \Big(1 + \frac{d_i}{A_i}\Big)^{s+1} + \frac{A_i^{s+1}}{s+1} \log\Big(1 + \frac{d_i}{A_i}\Big)\Big(1 + \frac{d_i}{A_i}\Big)^{s+1} \\
&\quad - \frac{A_i^{s+1}}{s+1} \log A_i + \frac{A_i^{s+1}}{(s+1)^2} - \frac{A_i^{s+1}}{(s+1)^2}\Big(1 + \frac{d_i}{A_i}\Big)^{s+1} \\
&= A_i^s \log A_i \, d_i + \frac{s}{2} A_i^{s-1} \log A_i \, d_i^2 + o\left(A_i^{s-1} \log A_i \, d_i^2\right). \qquad (2.6)
\end{aligned}$$

From (2.6) and Lemma 2.1 we find that

$$\begin{aligned}
\int_1^{A_n} x^s \log x \, dx &= \frac{A_n^{s+1}}{s+1} \log A_n - \frac{A_n^{s+1}}{(s+1)^2} + \frac{1}{(s+1)^2} \\
&= \int_1^{A_1} x^s \log x \, dx + \sum_{i=1}^{n-1} \int_{A_i}^{A_{i+1}} x^s \log x \, dx \\
&= \int_1^{A_1} x^s \log x \, dx + \sum_{i=1}^{n-1} A_i^s \; d_i \log A_i \\
&\quad + \left(\frac{s}{2} + o(1)\right) \sum_{i=1}^{n-1} A_i^{s-1} \; d_i^2 \log A_i,
\end{aligned}$$

that is,

$$\begin{aligned}
\frac{\log A_n}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{A_n^{s+1}} \frac{1}{(s+1)^2} &= \frac{1}{A_n^{s+1}} \int_1^{A_1} x^s \log x \, dx \\
+ \frac{1}{A_n^{s+1}} \sum_{i=1}^{n-1} A_i^s \; d_i \log A_i &+ \left(\frac{s}{2} + o(1)\right) \frac{1}{A_n^{s+1}} \sum_{i=1}^{n-1} A_i^{s-1} \; d_i^2 \log A_i. \qquad (2.7)
\end{aligned}$$

Equation (2.7) implies that the equation

$$\frac{1}{A_n^{s+1}}\sum_{i=1}^{n-1} A_i^{s-1}\ d_i^2\log A_i = o(1)$$

and the equation

$$\frac{\log A_n}{s+1}-\frac{1}{(s+1)^2}+o(1)=\frac{1}{A_n^{s+1}}\sum_{i=1}^{n-1} A_i^{s}\ d_i\log A_i$$

are equivalent. On the other hand the last equation is equivalent to the equation

$$\frac{\left(\prod_{i=1}^{n-1} A_i^{d_iA_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}}=(1+o(1))e^{-1/(s+1)^2}.$$

This completes the proof. □

In the next theorem we consider the case $s = 0$ for limit formula (2.1).

Theorem 2.3. *Let A_n be a strictly increasing sequence of positive real numbers tending to infinity satisfying the asymptotic formula $A_n \sim A_{n+1}$, and let $d_n = A_{n+1} - A_n$. Then*

$$\frac{\left(\prod_{i=1}^{n-1} A_i^{d_i}\right)^{1/A_n}}{A_n}\to\frac{1}{e}. \tag{2.8}$$

Proof. As in Theorem 2.2, by use of (2.3) (with $s = 0$) and (2.4), we obtain (compare with (2.6))

$$\int_{A_i}^{A_{i+1}}\log x\,dx = d_i\log A_i+\frac{1}{2}\frac{d_i^2}{A_i}+o\left(\frac{d_i^2}{A_i}\right). \tag{2.9}$$

From (2.9) (as in Theorem 2.2) we obtain (compare with (2.7))

$$\log A_n - 1+\frac{1}{A_n}=\frac{\int_1^{A_1}\log x\,dx}{A_n}+\frac{\sum_{i=1}^{n-1} d_i\log A_i}{A_n}+\frac{1}{2}\frac{\sum_{i=1}^{n-1}\frac{d_i^2}{A_i}}{A_n}+\frac{\sum_{i=1}^{n-1} o\left(\frac{d_i^2}{A_i}\right)}{A_n}. \tag{2.10}$$

Equation (2.10) implies that the equation

$$\frac{1}{2}\frac{\sum_{i=1}^{n-1}\frac{d_i^2}{A_i}}{A_n} + \frac{\sum_{i=1}^{n-1} o\left(\frac{d_i^2}{A_i}\right)}{A_n} = o(1), \tag{2.11}$$

and the equation

$$\log A_n - 1 = \frac{\sum_{i=1}^{n-1} d_i \log A_i}{A_n} + o(1) \tag{2.12}$$

are equivalent. That is, equation (2.11) holds if and only if equation (2.12) holds.

Now, equation (2.12) is equivalent to equation (2.8) and equation (2.11) is equivalent to the equation

$$\frac{\sum_{i=1}^{n-1}\frac{d_i^2}{A_i}}{A_n} = o(1). \tag{2.13}$$

Consider the two possible cases. That is, either the series of positive terms $\sum_{i=1}^{\infty}\frac{d_i^2}{A_i}$ converges or diverges. Now, $\frac{\frac{d_i^2}{A_i}}{d_i} = \frac{d_i}{A_i} \to 0$ and consequently by Lemma 2.1 we have

$$\frac{\sum_{i=1}^{n-1}\frac{d_i^2}{A_i}}{\sum_{i=1}^{n-1} d_i = A_n - A_1} \to 0,$$

that is, equation (2.13) holds. The theorem is proved. □

Now, we shall prove the following theorem, which is based on a stronger condition.

Theorem 2.4. *If in Theorem 2.2 the sequence A_n satisfies the stronger condition $\frac{d_n \log A_n}{A_n} \to 0$, then the limit formula* (2.1) *holds.*

Proof. The stronger condition can be written in the form

$$\frac{\frac{d_i^2 \log A_i}{A_i}}{d_i} \to 0.$$

Therefore by Lemma 2.1 we have

$$\frac{\sum_{i=1}^{n-1}\frac{d_i^2 \log A_i}{A_i}}{A_n} \to 0.$$

Finally

$$0 \leq \frac{\sum_{i=1}^{n-1} A_i^{s-1} d_i^2 \log A_i}{A_n^{s+1}} \leq \frac{\sum_{i=1}^{n-1} \frac{A_i^{s-1} d_i^2 \log A_i}{A_i^s}}{A_n} = \frac{\sum_{i=1}^{n-1} \frac{d_i^2 \log A_i}{A_i}}{A_n} \to 0,$$

that is, limit formula (2.2) and consequently limit formula (2.1) also holds. The theorem is proved. □

We therefore prove the following theorem that establishes a generalization of the Schaumberger's limit formula (1.2).

THEOREM 2.5. *Let A_n be a strictly increasing sequence of positive real numbers tending to infinity such that $\frac{d_n \log A_n}{A_n} \to 0$, where $d_n = A_{n+1} - A_n$. Then for any integer $s \geq 0$,*

$$\frac{\left(\prod_{i=1}^{n} A_i^{d_i A_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}} \to e^{-1/(s+1)^2}. \tag{2.14}$$

PROOF. It is an immediate consequence of Theorem 2.2, Theorem 2.3 and Theorem 2.4. Since

$$\begin{aligned} \frac{\left(\prod_{i=1}^{n} A_i^{d_i A_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}} &= \frac{\left(\prod_{i=1}^{n-1} A_i^{d_i A_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}} \left((A_n)^{d_n A_n^s}\right)^{1/A_n^{s+1}} \\ &= \frac{\left(\prod_{i=1}^{n-1} A_i^{d_i A_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}} e^{\frac{d_n \log A_n}{A_n}}, \end{aligned}$$

then (since $\frac{d_n \log A_n}{A_n} \to 0$)

$$\frac{\left(\prod_{i=1}^{n} A_i^{d_i A_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}} \sim \frac{\left(\prod_{i=1}^{n-1} A_i^{d_i A_i^s}\right)^{1/A_n^{s+1}}}{A_n^{1/(s+1)}}.$$

The theorem is proved. □

It can be seen that Theorem 2.5 with $A_n = n$ gives Schaumberger's limit formula (1.2), since in this case we have $\frac{d_n \log A_n}{A_n} = \frac{\log n}{n} \to 0$. Thus, limit in (1.1) is also obtained.

In the following theorem, we show that if a sequence A_n satisfies the hypothesis of Theorem 2.5, then limit formula (2.14) also holds for the sequence A_n^k, where $k \in \mathbb{N}$.

THEOREM 2.6. *Let A_n satisfies the hypothesis of Theorem* 2.5. *Then limit formula* (2.14) *holds for the sequence A_n^k, where $k \in \mathbb{N}$.*

PROOF. We know from the hypothesis of Theorem 2.5 that A_n is a strictly increasing sequence of positive real numbers tending to infinity such that $\frac{d_n \log A_n}{A_n} \to 0$. We must show that sequence A_n^k $(\forall k \in \mathbb{N})$ also holds under the hypothesis of Theorem 2.5.

Clearly A_n^k $(\forall k \in \mathbb{N})$ is also a strictly increasing sequence of positive real numbers tending to infinity. Now, we have the following well-known identity:

$$a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1}\right).$$

Therefore,

$$\begin{aligned} A_{n+1}^k - A_n^k &= d_n\left(A_{n+1}^{k-1} + A_{n+1}^{k-2}A_n + \cdots + A_{n+1}A_n^{k-2} + A_n^{k-1}\right) \\ &= (1+o(1))k d_n A_n^{k-1} \end{aligned}$$

and consequently

$$\begin{aligned} \frac{\left(A_{n+1}^k - A_n^k\right)\log A_n^k}{A_n^k} &= \frac{(1+o(1))k d_n A_n^{k-1} k \log A_n}{A_n^k} \\ &= \left((1+o(1))k^2 \frac{d_n \log A_n}{A_n}\right) \to 0. \end{aligned}$$

Hence, the sequence A_n^k $(\forall k \in \mathbb{N})$ satisfies the hypothesis of Theorem 2.5, therefore limit formula (2.14) also holds for A_n^k. The theorem is proved. □

Some well-known sequences follow the conditions of Theorem 2.5, consequently limit formula (2.14) applies to them. We show this in the following theorem and corollaries thereafter.

THEOREM 2.7. *Let A_n be a strictly increasing sequence of positive real numbers tending to infinity and let $d_n = A_{n+1} - A_n$. If $d_n < cA_n^\theta$, where c and $0 < \theta < 1$ are constants, then limit formula* (2.14) *holds.*

PROOF. Since A_n is a strictly increasing sequence of positive real numbers tending to infinity, we have immediately $\frac{d_n \log A_n}{A_n} \geq 0$ for $n \in \mathbb{N}$. Therefore

$$0 \leq \frac{d_n \log A_n}{A_n} < \frac{cA_n^\theta \log A_n}{A_n} = \frac{c \log A_n}{A_n^{1-\theta}} \to 0,$$

which gives $\frac{d_n \log A_n}{A_n} \to 0$. Hence, by Theorem 2.5 the limit formula (2.14) holds. The theorem is proved. □

COROLLARY 2.8. *Limit* (2.14) *holds for the sequence* p_n *of prime numbers.*

PROOF. It is an immediate consequence of Theorem 2.7, since it is well-known that (see, e.g., [1], [2]) there exist constants c and $0 < \theta < 1$ such that $d_n < cp_n^\theta$. The corollary is proved. □

COROLLARY 2.9. *Limit formula* (2.14) *holds for any sequence* A_n *such that* $d_n = A_{n+1} - A_n$ *is bounded. For example, for any linear sequence* $A_n = an + b$, *where* $a > 0$ *and* b *is integer. Also, limit formula* (2.14) *holds for any sequence in the form* $A_n = an^b$, *where* a *and* b *are positive integers, since* $\log A_n \sim b \log n$ *and by the binomial formula* $d_n \sim abn^{b-1}$. *Another example is the sequence* P_n *of perfect powers, since (see* [4]*)* $P_n \sim n^2$, $\log P_n \sim 2 \log n$ *and* $d_n = P_{n+1} - P_n < 2n$.

PROOF. Use Theorem 2.7. □

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