Annales Mathematicae Silesianae **34** (2020), no. 2, 286–304
DOI: 10.2478/amsil-2020-0007

Report of Meeting

The Twentieth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities Hajdúszoboszló (Hungary), January 29–February 1, 2020

The Twentieth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities was held in Hotel Aurum, Hajdúszoboszló, Hungary, from January 29 to February 1, 2020. It was organized by the Department of Analysis of the Institute of Mathematics of the University of Debrecen.[1]

The 30 participants came from the University of Silesia (Poland), the University of Debrecen (Hungary), the University of Rzeszów (Poland), the Pedagogical University of Cracow (Poland), the University of Zielona Góra (Poland), The John Paul II Catholic University of Lublin (Poland) and the Karlsruhe Institute of Technology (Germany), 10 from the first, 15 from the second and 1 from each of the other universities.

Professor Zsolt Páles opened the Seminar and welcomed the participants to Hajdúszoboszló. In his address, he proposed to dedicate the Twentieth DKWS to the memory of Professor János Aczél who passed away on January 1, 2020. Professor Aczél was the founder of the Department of Analysis of the University of Debrecen and did a pioneering work on the theory of functional equations. The dedication was enthusiastically accepted by all the participants.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iterative equations,

Received: 13.03.2020. Accepted: 20.04.2020. Published online: 22.06.2020.
[1]The Winter Seminar was supported by the Institute of Mathematics, University of Debrecen and by the project 2019-2.1.11-TÉT-2019-00049.

equations on algebraic structures, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity and Walsh–Fourier analysis. Interesting discussions were generated by the talks.

The closing address was given by Professor Maciej Sablik. His invitation to the Twenty-first Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities in January 2021 in Poland was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in Section 1, problems and remarks in chronological order in Section 2, and the list of participants in the final section.

1. Abstracts of talks

Anteneh Tilahun Adimasu: *Multi-parameter setting (C,α) means with respect to one dimensional Vilenkin system* (Joint work with György Gát)

We prove that the maximal operator of the (C, α_n)-means of the one dimensional Vilenkin-Fourier series is of weak type (L^1, L^1). Moreover, we prove the almost everywhere convergence of the (C, α_n) means of integrable functions (i.e. $\sigma_n^{\alpha_n} f \longrightarrow f$), where $n \in \mathbb{N}_{\alpha,q}$ and $n \longrightarrow \infty$ for $f \in L^1(G_m)$ (G_m is a bounded Vilenkin group), for every sequence $\alpha = (\alpha_n)$ and $0 < \alpha_n < 1$.

Roman Badora: *Stability results and separation theorems*

In the talk, we summarize relationships between stability results and separation theorems. We show that results regarding different types of stability of the Cauchy functional equation are equivalent to suitable types of theorems on separation (selection) by an additive map.

Mihály Bessenyei: *Generalized fractals: existence, uniqueness, dimension* (Joint work with Evelin Pénzes)

In his fundamental paper, Hutchinson established the unique existence of fractals defined by contractions of a complete metric space, and gave a formula for their Hausdorff dimension in terms of the contractions' factors. The aim of the talk is to present similar results for fractals determined by Matkowski–Browder-type contractions.

Zoltán Boros: *Regularity of monomials along monomials*

Adopting some arguments of [3], we establish the following lemma: if $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree $n \in \mathbb{N}$ and $2 \leq m \in \mathbb{N}$ such that the mapping

$$x \mapsto f(x^m) - x^{n(m-1)} f(x)$$

is bounded on an interval of positive length, then

$$f(x^m) = x^{n(m-1)} f(x) \tag{1}$$

holds for every $x \in \mathbb{R}$.

It was proved in [5], [1] and [2], respectively, that a generalized monomial f of degree $n \in \{1, 2, 3\}$ fulfills equation (1) for a fixed $2 \leq m \in \mathbb{N}$ if, and only if, f is given by $f(x) = cx^n \quad (x \in \mathbb{R})$ with an arbitrary real coefficient $c = f(1)$.

Combining these results, we obtain the following theorem: if $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree $n \in \{1, 2, 3\}$ and $2 \leq m \in \mathbb{N}$ such that the mapping

$$x \mapsto f(x^m) - x^{n(m-1)} f(x)$$

is bounded on an interval of positive length, then $f(x) = cx^n \quad (x \in \mathbb{R})$ with an arbitrary real coefficient c.

Motivated by the corresponding results in [2], [3], [4] and [5], we consider similar statements for exponents $-2 \geq m \in \mathbb{Z}$ as well.

References

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[3] Z. Boros and E. Gselmann, *Hyers-Ulam stability of derivations and linear functions*, Aequationes Math. **80** (2010), 13–25.

[4] E. Garda-Mátyás, *Quadratic functions fulfilling an additional condition along hyperbolas or the unit circle*, Aequationes Math. **93** (2019), 451–465.

[5] A. Nishiyama and S. Horinouchi, *On a system of functional equations*, Aequationes Math. **1** (1968), 1–5.

Pál Burai: *Limit theorems for Bajraktarević and Cauchy quotient means of independent identically distributed random variables* (Joint work with Mátyás Barczy)

We derive strong law of large numbers and central limit theorems for Bajraktarević, Gini and exponential (also called Beta-type) and logarithmic

Cauchy quotient means of independent identically distributed (i.i.d.) random variables. The exponential and logarithmic Cauchy quotient means of a sequence of i.i.d. random variables behave asymptotically normal with the usual square root scaling just like the geometric means of the given random variables. Somewhat surprisingly, the multiplicative Cauchy quotient means of i.i.d. random variables behave asymptotically in a rather different way: in order to get a non-trivial normal limit distribution a time dependent centering is needed.

Jacek Chudziak: *Positive homogeneity of the principle of equivalent utility*

A premium principle is a way of assigning to every risk a non-negative real number, which is interpreted as a premium for insuring the risk. We deal with the principle of equivalent utility. The principle is based on the assumption that a premium for a given risk is established in such a way that an insurance company is indifferent between rejecting the contract and entering into it. Inspired by the results in [1] we prove a characterization of positive homogeneity of the principle of equivalent utility under Cumulative Prospect Theory, one of the behavioral models of decisions made under risk.

Reference

[1] M. Kałuszka and M. Krzeszowiec, *Pricing insurance contracts under Cumulative Prospect Theory*, Insurance Math. Econom. **50** (2012), 159–166.

Borbála Fazekas: *Numerical solutions of second order partial differential equations using Walsh-Fourier series*

Walsh functions $\omega_0, \omega_1, \dots$ are step functions with range $\{-1, 1\}$ defined on the interval $[0, 1[$. More precisely they are given via

$$\omega_n(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}, \quad \text{with } n \in \mathbb{N}, n = \sum_{k=0}^{\infty} n_k 2^k,$$
$$x \in [0, 1[, \quad x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}.$$

We are looking for approximate solutions of two dimensional partial differential equations in the finite dimensional space $L_n = \mathcal{L}\big(\omega_i(x) \cdot \omega_j(t) \colon i, j = 0, 1, \dots, 2^n - 1\big)$.

György Gát: *Convergence and divergence properties of Cesàro means with varying parameters of Walsh-Fourier series*

Let $\alpha = (\alpha_n)$ be a sequence of reals, where $0 \le \alpha_n \le 1$ for every $n \in \mathbb{N}$. Let $\hat{f}(k) := \int_0^1 f(x)\omega_k(x)dx$ be the kth Walsh-Fourier coefficient of the integrable function f and define the (C, α) (varying parameter Cesàro) means of Walsh-Fourier series of f as

$$\sigma_n^{\alpha_n} f := \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n} A_{n-j}^{\alpha_n} \hat{f}(j)\omega_j,$$

where $A_n^\beta := \frac{(1+\beta)\cdots(n+\beta)}{n!}$ for parameter $\beta \in \mathbb{R} \setminus \{-1, -2, \dots\}$. It is well-known, that for $\alpha_n = 1$ (for all n) we have the Fejér means and the a.e. relation $\sigma_n^1 f \to f$. Meanwhile, for $\alpha_n = 0$ (for every n), $\sigma_n^0 f$ is the nth partial sum of the Walsh-Fourier series of function f for what there exists a negative result, i.e. an integrable function f such as $\sigma_n^0 f \to f$ nowhere. These varying parameter Cesàro means are introduced and firstly investigated by Akhobadze [1] in the case of the trigonometric system. He gave some approximation results corresponding to continuous functions. But no pointwise convergence result was given.

In this talk we show some recent almost everywhere convergence and divergence results [2, 3] for means of the above kind with respect to Walsh-Fourier series of integrable functions.

References

[1] T. Akhobadze, *On the convergence of generalized Cesàro means of trigonometric Fourier series. I*, Acta Math. Hungar. **115** (2007), 59–78.

[2] Gy. Gát and U. Goginava, *Maximal operators of Cesàro means with varying parameters of Walsh-Fourier series*, Acta Math. Hungar. **159** (2019), 653–668.

[3] Gy. Gát and U. Goginava, *Almost everywhere convergence and divergence of Cesàro means with varying parameters of Walsh-Fourier series*, Mat. Sb. To appear.

Roman Ger: *On alienation of two functional equations of quadratic type*

We deal with an alienation problem for an Euler-Lagrange type functional equation

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y)$$

assumed for fixed nonzero real numbers $a, b \notin \{-1, 1\}$, $a^2 \neq b^2$, and the classic quadratic functional equation

$$g(x + y) + g(x - y) = 2g(x) + 2g(y).$$

We were inspired by papers of Chang Il Kim, Giljun Han & Seong-A. Shim [2] and M.Eshaghi Gordji and H. Khodaei [1], where the special case $g = cf$ has been examined.

References

[1] M. Eshaghi Gordji and H. Khodaei, *On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations*, Abstr. Appl. Anal. **2009**, Art. ID 923476, 11 pp.

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Attila Gilányi: *On equations, inequalities and the Dinghas interval derivative*

The nth-order Dinghas derivative (introduced by Alexander Dinghas in [1]) of a real valued function f at a point $\xi \in \mathbb{R}$ is defined by

$$D^n f(\xi) = \lim_{\substack{\alpha\le\xi\le\beta \\ \beta-\alpha\searrow 0}} \left(\frac{-n}{\beta-\alpha}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\left(1-\frac{k}{n}\right)\alpha + \frac{k}{n}\beta\right),$$

if the limit exists. In his paper [1], Dinghas also proved that, assuming n-times differentiability in the common sense for f at ξ, the limit above exists and it coincides with the nth derivative of f. However, the class of functions differentiable in the sense of Dinghas is considerably wider than the family of functions differentiable in the classical setting.

In this talk, we overview some results connected to well-known classes of functional equations, inequalities and the operator above obtained during the last twenty years.

Reference

[1] A. Dinghas, *Zur Theorie der gewöhnlichen Differentialgleichungen*, Ann. Acad. Sci. Fenn. Ser. A I No. **375** (1966), 19 pp.

Angshuman R. Goswami: *Properties of* Φ*-monotone and* Φ*-Hölder functions* (Joint work with Zsolt Páles)

A real valued function f defined on a real open interval I is called Φ-monotone if, for all $x, y \in I$ with $x \le y$ it satisfies

$$f(x) \le f(y) + \Phi(y-x),$$

where $\Phi : [0, \ell(I)[\to \mathbb{R}_+$ is a given nonnegative error function, where $\ell(I)$ denotes the length of the interval I. If f and $-f$ are simultaneously Φ-monotone, then f is said to be a Φ-Hölder function.

We can describe structural properties of these function classes, determine the error function which is the most optimal one. We show that optimal error functions for Φ-monotonicity and Φ-Hölder property must be subadditive and absolutely subadditive, respectively. Then we offer a precise formula for the lower and upper Φ-monotone and Φ-Hölder envelopes. We also introduce a generalization of the classical notion of total variation and we prove an extension of the Jordan decomposition theorem known for functions of bounded total variation.

Eszter Gselmann: *Remarks on the notion of homo-derivations* (Joint work with Gergely Kiss)

The purpose of this talk is to study the (different) notions of homo-derivations. These are additive mappings f of a ring R that also fulfill the identity

$$f(xy) = f(x)y + xf(y) + f(x)f(y) \qquad (x, y \in R)\,,$$

or (in case of the other notion) the system of equations

$$\begin{aligned} f(xy) &= f(x)f(y) \\ f(xy) &= f(x)y + xf(y) \end{aligned} \qquad (x, y \in R)\,.$$

Our primary aim is to investigate the above equation *without additivity* as well as its Pexiderized version, that is,

$$f(xy) = h(x)h(y) + xk(y) + k(x)y.$$

The results obtained show that in case of fields only trivial homo-derivations do exist, even without the additivity assumption.

Wojciech Jabłoński: *Families of additive iterative roots of identity and Hamel bases*

Let X be a real topological vector space and define

$$\mathcal{A}_X = \{a \in X^X : a \ \text{ is additive}\},$$
$$\mathcal{A}_X^{(q)} = \{a \in \mathcal{A}_X : q(a) = 0\},$$

where $q \in \mathbb{Q}[t]$ with $\deg q \geq 2$ is fixed. We discuss results (see [1], [3], [4], [5]) concerning topological properties of families

$$\mathcal{A}_1 = \big\{a \in \mathcal{A}_X^{(q)} : a \ \text{ is discontinuous and } \ a(\mathcal{H}) \setminus \mathcal{H} \neq \emptyset \ \text{ for every infinite set } \mathcal{H} \subset X \ \text{ of vectors linearly independent over } \mathbb{Q}\big\},$$

$$\mathcal{A}_2 = \big\{a \in \mathcal{A}_X^{(q)} : a \text{ is discontinuous and } a(\mathcal{H}) = \mathcal{H} \text{ for some Hamel basis } \mathcal{H} \subset X\big\}.$$

Our approach bases on tools from [2] and [6].

References

[1] K. Baron, *On additive involutions and Hamel bases*, Aequationes Math. **87** (2014), 159–163.

[2] K. Baron and P. Volkmann, *Dense sets of additive functions*, Seminar LV, No. **16** (2003), 4 pp. http://www.math.us.edu.pl/smdk.

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[5] W. Jabłoński, *A characterization of additive iterative roots of identity with respect to invariant subspaces*, Aequationes Math. **93** (2019), 247–255.

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Justyna Jarczyk: *On a functional equation appearing on the margins of a mean invariance problem* (Joint work with Witold Jarczyk)

Given a continuous strictly monotonic real-valued function α, defined on an interval I, and a function $\omega : I \to (0, +\infty)$ we denote by B_ω^α the Bajraktarević mean generated by α and weighted by ω:

$$B_\omega^\alpha(x, y) = \alpha^{-1}\left(\frac{\omega(x)}{\omega(x)+\omega(y)}\alpha(x) + \frac{\omega(y)}{\omega(x)+\omega(y)}\alpha(y)\right), \quad x, y \in I.$$

We find the formula for all possible three times differentiable solutions (φ, ψ) of the functional equation

$$r(x)B_s^\varphi(x, y) + r(y)B_t^\psi(x, y) = r(x)x + r(y)y,$$

where $r, s, t : I \to (0, +\infty)$ are three times differentiable functions and the first derivative of r does not vanish. However, we show that not every pair (φ, ψ) given by the found formula actually satisfies the above equation.

Tibor Kiss: *Balancing property of a subclass of iteratively quasi-arithmetic means*

A two-variable mean $M : I \times I \to \mathbb{R}$ is called *balanced* if, for any $x, y \in I$, the equality

$$M\big(M(x, u), M(u, y)\big) = u$$

holds with $u := M(x,y)$. The archetypal examples for such means are quasi-arithmetic means, but the above equation has pathological solutions as well. In general, it is an interesting question that, having a two-variable mean, beside the balancing property, what we need to assume to conclude that it is quasi-arithmetic. This question was investigated by several authors concerning different classes of means. In each case some level of differentiability of the mean was assumed.

The aim of the talk is to treat the above equation in the class of iteratively quasi-arithmetic means avoiding any differentiability assumption.

Radosław Łukasik: *A functional equation with biadditive functions*

The functional equation

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle, \quad x,y \in H, \tag{1}$$

where H, K are unitary spaces, $f, g\colon H \to K$, was studied in several papers ([1], [3], [5], [6]).

In [4] authors give a natural generalization of such functional equations in the case of abelian groups. They consider biadditive mappings instead of inner products.

In [2] we can find a different approach. Instead of taking two different functions on the left side of (1), we change only the right side of (1), so we obtain

$$\langle f(x)|f(y)\rangle = \langle x|g(y)\rangle\,, \quad x,y \in X,$$

with two unknown functions $f\colon X \to Y$, $g\colon X \to X$.

In this talk we generalize the above equation - we consider biadditive mappings instead of inner products.

References

[1] R. Łukasik, *A note on the orthogonality equation with two functions*, Aequationes Math. **90** (2016), 961–965.
[2] R. Łukasik, *Some orthogonality equation with two functions*, Miskolc Math. Notes **18** (2017), 953–960.
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[5] M.M. Sadr, *Decomposition of functions between Banach spaces in the orthogonality equation*, Aequationes Math. **91** (2017), 739–743.
[6] P. Wójcik, *On an orthogonality equation in normed spaces*, Funct. Anal. Appl. **52** (2018), 224–227.

Janusz Morawiec: *Iterative functional equations and medial limits*

We will discuss how to apply medial limits to iterative functional equations.

Gergő Nagy: *Homomorphisms with respect to operator means induced by the functional calculus*

In this talk, we investigate maps on sets of positive operators which stem from the continuous functional calculus and transform a Kubo-Ando operator mean σ into another τ. We establish that under quite mild conditions, a mapping ϕ can have this property only in the trivial case, i.e. when σ and τ are nontrivial weighted harmonic means and ϕ is induced by a function which is of the form "constant $\times$ the generating function of a mean of the latter kind". In the setting where exactly one of σ and τ is a weighted arithmetic mean, we show that under fairly weak assumptions, the mentioned transformer property never holds. Finally, when both of σ and τ are such a mean, it turns out that the latter property is satisfied only in the trivial case, i.e. for maps induced by affine functions.

Andrzej Olbryś: *On separation by functions of bounded variation*

In our talk we are looking for the conditions under which two functions defined on a compact interval can be separated by a function of bounded variation.

Zsolt Páles: *Hardy type inequalities for weighted means with best constants* (Joint work with Paweł Pasteczka)

Using recent results concerning the homogenization [4] and the Hardy property of weighted means [3], we establish sharp Hardy constants for concave and monotone weighted quasideviation means and for a few particular subclasses of this broad family. More precisely, for a mean M like above and a sequence (λ_n) of positive weights such that $\lambda_n/(\lambda_1+\dots+\lambda_n)$ is nondecreasing, we determine the smallest number $H \in (1, +\infty]$ such that

$$\sum_{n=1}^{\infty} \lambda_n M\big((x_1,\dots,x_n),(\lambda_1,\dots,\lambda_n)\big) \le H \cdot \sum_{n=1}^{\infty} \lambda_n x_n \quad \text{for all } x \in \ell_1(\lambda).$$

It turns out that H depends only on the limit of the sequence $(\lambda_n/(\lambda_1+\dots+\lambda_n))$ and the behaviour of the mean M near zero. The results obtained in some earlier papers [1,2] will be extended and generalized (see [5]).

References

[1] Zs. Páles and P. Pasteczka, *Characterization of the Hardy property of means and the best Hardy constants*, Math. Inequal. Appl. **19** (2016), 1141–1158.
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[5] Zs. Páles and P. Pasteczka, *On Hardy type inequalities for weighted quasideviation means*, Math. Inequal. Appl. To appear.

Maciej Sablik: *Non-symmetric functional equations stemming from MVT*

Again, we go back to questions considered a year ago during the 19th Winter Seminar. The study started in [2] and concerns the equation

$$\frac{f(x)-f(y)}{x-y} = g(\Phi(x,y)),$$

where $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a piecewise linear mapping, and both f and g are unknown. The problem has arised in connection with the paper [1].

References

[1] P. Carter and D. Lowry-Duda, *On functions whose mean value abscissas are midpoints, with connections to harmonic functions*, Amer. Math. Monthly **124** (2017), 535–542.
[2] M. Sablik, *An elementary method of solving functional equations*, Ann. Univ. Sci. Budapest. Sect. Comput. **48** (2018), 181–188.

Justyna Sikorska: *Alienation and the stability problem*

The alienation phenomenon for the two Cauchy equations

$$f(x+y) = f(x) + f(y),$$

$$g(x+y) = g(x)g(y)$$

was investigated by Roman Ger in [1]. We combine this notion with the study of the stability problem.

Reference

[1] R. Ger, *Additivity and exponentiality are alien to each other*, Aequationes Math. **80** (2010), 111–118.

Patricia Szokol: *Hermite-Hadamard type inequality for certain Schur convex functions* (Joint work with Pál Burai, Judit Makó)

In the presentation we will investigate symmetric, continuous functions whose domain is in $\mathbb{R}^n$ that satisfy a Hermite-Hadamard type inequality. In the main result we prove that such functions are necessarily Jensen-convex. We also present a Korovkin-type approximation theorem, which plays the key role in the proof of our main theorem.

Tomasz Szostok: *Some results around the Hermite-Hadamard inequality*

In this talk we present results of Hermite-Hadamard type for some functions which are not necessarily convex as well as inequalities which are true for convex functions only if these functions satisfy certain additional assumptions.

Peter Volkmann: *Comparison theorem for functional equations*

A general result will be presented, which can be applied to integral equations in ordered topological vector spaces.

Amr Zakaria: *On the equality of Cauchy means to quasiarithmetic means* (Joint work with Rezső Lovas and Zsolt Páles)

The goal of this talk is to characterize the equality of Cauchy mean (cf. [2]) to a two-variable quasiarithmetic mean (cf. [1]) without additional unnatural regularity assumptions. More precisely, we provide six necessary and sufficient conditions to the above mentioned equality problem. One of these conditions says that a Cauchy mean is quasiarithmetic if and only if the range of its generating functions is covered by a non-degenerate conic section.

References

[1] G.H. Hardy and J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934 (first edition), 1952 (second edition).
[2] E.B. Leach and M.C. Sholander, *Multivariable extended mean values*, J. Math. Anal. Appl. **104** (1984), 390–407.

Thomas Zürcher: *Regularity of solutions of some functional equations studied by Janusz Matkowski and Kazimierz Nikodem*

In this talk, we will review solutions of the functional equation

$$\varphi = \sum_{k=1}^{n} g_k \varphi \circ f_k + h.$$

We are looking for solutions φ in Lebesgue spaces under appropriate conditions on f_k, g_k, and h. Earlier works on this equation are [1] and [4], and our papers are [2] and [3].

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[3] J. Morawiec and T. Zürcher, *Some classes of linear operators involved in functional equations*, Ann. Funct. Anal. **10** (2019), 381–394.
[4] K. Nikodem, *On ϵ-invariant measures and a functional equation*, Czechoslovak Math. J. **41(116)** (1991), 565–569.

Marcin Zygmunt: *Sums of periodic functions*

One of the problems proposed for the last students competition ISTCM 2019 in Katowice was to write, if possible, the identity function from $\mathbb{R}$ to $\mathbb{R}$ as a sum of periodic functions. Namely, the problem can be reformulated in the following form: find two periodic functions f and g satisfying the functional equation $f(x) + g(x) = x$ for all real x (cf. [1]).

It is clear that the sum of finitely many *continuous* periodic functions is bounded, thus no nonconstant polynomial can be written in this form. But, as we will see, if the functions are not required to be continuous, then the previous statement is no longer valid. Moreover, such periodic functions are also additive (cf. [2]).

More general result is also true: every polynomial of degree $n > 0$ in one real variable can be expressed as the sum of $n + 1$ (but not of n) periodic functions on $\mathbb{R}$. Moreover, there exist functions which cannot be expressed as a sum of finitely many periodic functions. Examples of such functions will be presented in the second part of the talk.

References

[1] http://istcim.math.us.edu.pl/ISTCM_2019_problems.pdf, Katowice (2019).
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2. Problems and remarks

1. Remark (*Bajraktarević means via Chebyshev systems)* Let I be an open interval and let $(f, g) : I \to \mathbb{R}^2$ be a 2-dimensional Chebyshev system, i.e., let f, g be continuous functions satisfying

$$\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} \neq 0 \tag{1}$$

for all $x, y \in I$ with $x \neq y$.

An important particular setting is when g is a positive and f/g is a strictly monotone function. Then the identity

$$\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} = g(x)g(y)\left(\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(y)\right)$$

shows that (f,g) is a 2-dimensional Chebyshev system.

Define the graph $\Gamma_{f,g} \subseteq \mathbb{R}^2$ of a 2-dimensional Chebyshev system (f,g) by

$$\Gamma_{f,g} := \{(f(x),g(x)) \mid x \in I\}.$$

For the construction of a mean generated by a 2-dimensional Chebyshev system, we need the following

LEMMA. *Let $(f,g) : I \to \mathbb{R}^2$ be a 2-dimensional Chebyshev system and let (p,q) be an arbitrary point of the convex hull of the graph $\Gamma_{f,g}$. Then there exists a unique pair $(x,t) \in I \times \mathbb{R}_+$ such that*

$$p = tf(x) \qquad \textit{and} \qquad q = tg(x). \tag{2}$$

PROOF. First we prove the uniqueness. Assume that, for some $(x,t),(y,s) \in I \times \mathbb{R}_+$, we have

$$p = tf(x) = sf(y) \qquad \text{and} \qquad q = tg(x) = sg(y).$$

Then (t,s) is a nontrivial solution of the homogeneous linear system of equations

$$tf(x) - sf(y) = 0 \qquad \text{and} \qquad tg(x) - sg(y) = 0.$$

Therefore, the base determinant of this system vanishes, which, by the Chebyshev property, implies $x = y$. The pair $(f(x),g(x))$ cannot be equal to $(0,0)$ (otherwise (1) is not valid for $y \neq x$). Thus the equality $t(f(x),g(x)) = s(f(x),g(x))$ yields that $s = t$.

To prove the existence, we shall apply a result of the paper [2] which says that, for any 2-dimensional Chebyshev system (f,g), there exist four real constants a,b,c,d with $ad \neq bc$ such that, for the functions $f^* := af + bg$ and $g^* := cf + dg$, we have that g^* is positive and f^*/g^* is strictly increasing.

Let (p,q) be an arbitrary element from the convex hull of $\Gamma_{f,g}$. Then $(p^*,q^*) := (ap + bq, cp + dq)$ is in the convex hull of Γ_{f^*,g^*}. Therefore, for some n, for some elements $x_1, \dots, x_n$ of the interval I and for some positive weights $t_1, \dots, t_n$ with $t_1 + \dots + t_n = 1$, we have

$$(p^*, q^*) := t_1(f^*(x_1), g^*(x_1)) + \dots + t_n(f^*(x_n), g^*(x_n)).$$

Thus, $q^* > 0$ and, by the convexity of the codomain $\left(\frac{f^*}{g^*}\right)(I)$,

$$\frac{p^*}{q^*} = \sum_{k=1}^{n} \frac{t_k g^*(x_k)}{t_1 g^*(x_1) + \dots + t_n g^*(x_n)} \cdot \left(\frac{f^*}{g^*}\right)(x_k) \in \left(\frac{f^*}{g^*}\right)(I),$$

which shows that $\frac{p^*}{q^*}$ belongs to the range of the strictly increasing function f^*/g^*. Therefore, we may define $x \in I$ and $t > 0$ by

$$x := \left(\frac{f^*}{g^*}\right)^{-1}\left(\frac{p^*}{q^*}\right) \qquad \text{and} \qquad t := \frac{q^*}{g^*(x)}.$$

The second equality yields that $q^* = tg^*(x)$, and then the first one implies that $p^* = q^*(f^*(x)/g^*(x)) = tf^*(x)$. Thus,

$$ap + bq = t(af(x) + bg(x)) \qquad \text{and} \qquad cp + dq = t(cf(x) + dg(x)).$$

Using that $ad \neq bc$, it easily follows from here that the equalities in (2) must be valid. □

Finally, we are in the position to define a weighted mean generated by a 2-dimensional Chebyshev system. Given n elements $x_1, \dots, x_n$ of the interval I and positive weights $t_1, \dots, t_n$ with $t_1 + \dots + t_n = 1$, the point

$$(p, q) := t_1(f(x_1), g(x_1)) + \dots + t_n(f(x_n), g(x_n))$$

is in the convex hull of the graph $\Gamma_{f,g}$. Thus, in view of the above Lemma, there exists a unique pair $(x, t) \in I \times \mathbb{R}_+$ such that

$$(3)\quad t_1 f(x_1) + \dots + t_n f(x_n) = tf(x) \quad \text{and} \quad t_1 g(x_1) + \dots + t_n g(x_n) = tg(x).$$

The unique value x so defined is called the *generalized weighted Bajraktarević mean* of $x_1, \dots, x_n$ with weights $t_1, \dots, t_n$ and it is denoted by

$$B_{f,g}\begin{pmatrix} x_1 & \dots & x_n \\ t_1 & \dots & t_n \end{pmatrix}.$$

In the particular case when g is positive and f/g is strictly monotone, we can solve the system of equations (3) with respect to x. Indeed, divide the first equality by the second equality in (3) side by side. Then we obtain that

$$\frac{t_1 f(x_1) + \cdots + t_n f(x_n)}{t_1 g(x_1) + \cdots + t_n g(x_n)} = \frac{f(x)}{g(x)},$$

which implies that

$$B_{f,g}\begin{pmatrix} x_1 & \dots & x_n \\ t_1 & \dots & t_n \end{pmatrix} = \left(\frac{f}{g}\right)^{-1}\left(\frac{t_1 f(x_1) + \cdots + t_n f(x_n)}{t_1 g(x_1) + \cdots + t_n g(x_n)}\right).$$

This is one of the standard forms of Bajraktarević means which were introduced in the paper [1].

The advantage of the above definition of Bajraktarević means is that the identity $B_{f,g} = B_{af+bg,cf+dg}$ (where $a, b, c, d \in \mathbb{R}$ with $ad \neq cb$) is an easy consequence.

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2. Remark (*Open problem on linear functional inequalities*) We deal here with inequalities of the form

$$\sum_{i=1}^{n} a_i f(\alpha_i x + (1-\alpha_i)y) \geq 0 \tag{1}$$

where $a_i \in \mathbb{R}, \alpha_i \in [0,1]$ are given numbers. If we want to solve (1) in the class of continuous functions then we can proceed in the following way. First we find the number k such that the functions $x \mapsto x^i, i = 0, \dots, k$ satisfy (1) but the function $x \mapsto x^{k+1}$ does not. Then, from the main result of [1], we know that a continuous solution of (1) must be k-convex.

Now, we have to check if all k-convex functions indeed satisfy our equation. The most convenient way to check it is to use the so-called generalized Levin-Stechkin theorem, see [2], [3]. If in a concrete case the condition from this theorem is satisfied then we obtain the solution of (1) in the class of continuous functions. Namely, our equation is satisfied by a continuous function if and only if f is k-convex.

Therefore, a natural question may be asked if the procedure described above may be used to obtain solutions of (1) without the continuity assumption. Thus we present the following conjecture.

Conjecture. *Let* $a_i \in \mathbb{R}, \alpha_i \in [0,1] \cap \mathbb{Q}$ *be given numbers. If* (1) *is satisfied by a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *if and only if* f *is* k*-convex, then* $\Delta_h^{k+1} f(x) \geq 0, x, h \in \mathbb{R}$.

Note that in the conjecture it is assumed that the weights α_i are rational. Without this assumption, the conjecture is not true because it is known that, in general, t-Wright convexity does not imply Jensen's convexity.

It seems that any result toward proving (or disproving) this conjecture will be very useful in the theory of functional inequalities.

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