Annales Mathematicae Silesianae **34** (2020), no. 1, 81–95
DOI: 10.2478/amsil-2020-0018

ON THE BOREL CLASSES OF SET-VALUED MAPS OF TWO VARIABLES

Ľubica Holá, Grażyna Kwiecińska

Dedicated to Professor Zygfryd Kominek on his 75th birthday

Abstrakt. Using the Borel classification of set-valued maps, we present here some new results on set-valued maps which are similar to some of the well known theorems on functions due to Lebesgue and Kuratowski. We consider set-valued maps of two variables in perfectly normal topological spaces. It was proved in [11] that a set-valued map lower semicontinuous (i.e. of lower Borel class 0) in the first and upper semicontinuous (i.e. of upper Borel class 0) in the second variable is of upper Borel class 1 and also (with stronger assumptions) of lower Borel class 1. This result cannot be generalized into higher Borel classes. In this paper we show that a set-valued map of the upper (resp. lower) Borel class α in the first and lower semicontinuous and upper quasicontinuous (upper semicontinuous and lower quasicontinuous) in the second variable is of the lower (resp. upper) Borel class $\alpha + 1$. Also other cases are considered.

1. Introduction

Kuratowski in [9] introduced Borel classification of set-valued maps. The lower and the upper Borel classes of set-valued maps of one variable have been studied intensively among others by Brisac ([1]), Ewert ([2]), Garg ([4]),

Received: 14.01.2020. Accepted: 28.06.2020. Published online: 16.07.2020.
(2010) Mathematics Subject Classification: 54C60, 54C08, 28B20.
Key words and phrases: continuity, lower (upper) semicontinuous, lower (upper) quasicontinuous set-valued map, Borel classes of set-valued maps.

Hansell ([5]) and Kuratowski ([9, 10]), where many of the known results on real functions and also on lower and upper semicontinuous set-valued maps have been extended to the general Borel classes of set-valued maps.

Obviously each set-valued map of two variables may be treated as a set-valued map of a single variable. But in this case we have the possibility of formulation of hypotheses concerning a set-valued map in terms of its sectionwise properties.

The following result of Lebesgue is well known.

Theorem 1.1. *A function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the Borel class $\alpha + 1$ whenever it is of the Borel class α in the first and continuous in the second variable.*

This theorem was extended into the case of functions in metric spaces ([8]). We extend this result into set-valued case in more general spaces.

We will investigate set-valued maps in perfectly normal topological spaces. We show that a set-valued map of the upper (resp. lower) Borel class α in the first and lower semicontinuous and upper quasicontinuous (upper semicontinuous and lower quasicontinuous) in the second variable is of the lower (resp. upper) Borel class $\alpha + 1$. It turns out that if we replace assumptions on a set-valued map in the second variable by the upper (resp. lower) semicontinuity, then the set-valued map may not belong to any Borel class. We demonstrate some reinforcement of the lower semicontinuity with additional assumptions which do secure the lower Borel class $\alpha + 2$ of the set-valued map.

2. Preliminaries

We begin with a conventional notation and basic definitions. The sets of positive integers, real numbers and positive real numbers will be denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively.

Let T and Z be two nonempty sets. By a set-valued map from T to Z we mean a map which assigns to every point of T a nonempty subset of Z. If Φ is a set-valued map from T to Z, we denote it by $\Phi\colon T \rightsquigarrow Z$.

If $G \subset Z$, then two inverse images of G under Φ are defined as follows:

$$\Phi^-(G) = \{t \in T : \Phi(t) \cap G \neq \emptyset\} \quad \text{and} \quad \Phi^+(G) = \{t \in T : \Phi(t) \subset G\}.$$

One sees immediately that

$$\Phi^-(G) = T \setminus \Phi^+(Z \setminus G) \quad \text{and} \quad \Phi^+(G) = T \setminus \Phi^-(Z \setminus G).$$

Any function $\phi\colon T \to Z$ may be considered as a set-valued map assigning to $t \in T$ the singleton $\{\phi(t)\}$. In this case we have $\phi^-(G) = \phi^+(G) = \phi^{-1}(G)$.

Now let $(T, \mathcal{T}(T))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. Then $\mathcal{B}(T)$ will denote the σ-field of Borel subsets of T and $\mathcal{O}(t)$ the family of all open neigbourhoods of $t \in T$. Moreover $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ will denote the closure and the interior of $A \subset T$, respectively.

A set-valued map $\Phi\colon T \rightsquigarrow Z$ is lower (resp. upper) semicontinuous at a point $t \in T$, if for every open set V such that $\Phi(t) \cap V \neq \emptyset$ (resp. $\Phi(t) \subset V$), there exists $U(t) \in \mathcal{O}(t)$ such that

$$\Phi(s) \cap V \neq \emptyset \quad (\text{resp. } \Phi(s) \subset V) \quad \text{for every } s \in U(t).$$

Φ is lower (resp. upper) semicontinuous if it is lower (resp. upper) semicontinuous at each point of T. Φ is continuous if it is both lower and upper semicontinuous.

Obviously

(2.1) Φ is lower (resp. upper) semicontinuous if and only if $\Phi^-(G) \in \mathcal{T}(T)$ (resp. $\Phi^+(G) \in \mathcal{T}(T)$) whenever $G \in \mathcal{T}(Z)$.

In our paper we will also work with the notion of quasicontinuity for set-valued maps.

The notion of lower quasicontinuity and upper quasicontinuity for set-valued maps was introduced by Neubrunn in [13]. The notion of quasicontinuity for real functions of real variables was introduced by Kempisty in 1932 in [7].

A set-valued map $F\colon T \rightsquigarrow Z$ is lower (resp. upper) quasicontinuous at a point $t \in T$, if for every open set V in Z with $F(t) \cap V \neq \emptyset$ (resp. $F(t) \subset V$) and every $U(t) \in \mathcal{O}(t)$ there is a nonempty open set $G \subset U(t)$ such that

$$F(s) \cap V \neq \emptyset \ (\text{resp. } F(s) \subset V) \quad \text{for every } s \in G.$$

A set-valued map $F\colon T \rightsquigarrow Z$ is lower (resp. upper) quasicontinuous if it is lower (resp. upper) quasicontinuous at each point of T.

Let (Z, ρ) be a metric space. By $B(z, r)$ we denote an open ball centered in $z \in Z$ with radius $r > 0$.

For $z \in Z$ and $\Phi\colon T \rightsquigarrow Z$ we define the function $g_z\colon T \to \mathbb{R}$ by

$$g_z(t) = \rho(z, \Phi(t)).$$

The following properties are true (see [6]).

(2.2) If $\Phi\colon T \rightsquigarrow Z$ is a set-valued map, then

(i) Φ is lower semicontinuous iff g_z is upper semicontinuous for every $z \in Z$.

(ii) If Φ is upper semicontinuous, then g_z is lower semicontinuous for each $z \in Z$.

A function $f\colon T \to \mathbb{R}$ is said to be upper (resp. lower) quasicontinuous at $t \in T$ ([3]) if for every $\varepsilon > 0$ and for every $U(t) \in \mathcal{O}(t)$ there is a nonempty open set $G \subset U(t)$ such that $f(v) < f(t) + \varepsilon$ ($f(v) > f(t) - \varepsilon$) for every $v \in G$. A function $f\colon T \to \mathbb{R}$ is upper (resp. lower) quasicontinuous if it is upper (resp. lower) quasicontinuous at every $t \in T$.

The following properties are easy to verify.

(2.3) If $\Phi\colon T \rightsquigarrow Z$ is a set-valued map, then

(i) Φ is lower quasicontinuous iff g_z is upper quasicontinuous for every $z \in Z$.

(ii) If Φ is upper quasicontinuous, then g_z is lower quasicontinuous for each $z \in Z$.

Given any countable ordinal number $\alpha < \Omega$, we denote by $\sum_\alpha(T)$ (resp. $\prod_\alpha(T)$) the additive (resp. multiplicative) class α of the Borel subsets of T.

We follow [9] in assuming that $\Phi\colon T \rightsquigarrow Z$ is of the lower (resp. upper) Borel class α (lower or upper class α for short) if $\Phi^-(G) \in \sum_\alpha(T)$ (resp. $\Phi^+(G) \in \sum_\alpha(T)$) for each open set $G \subset Z$.

We will say that $\Phi\colon T \rightsquigarrow Z$ is $\mathcal{B}$-measurable if Φ is of the lower Borel class α or of the upper Borel class α.

Note that

(2.4) Φ is of the lower (resp. upper) class α if and only if $\Phi^+(D) \in \displaystyle\prod_\alpha(T)$ (resp. $\Phi^-(D) \in \displaystyle\prod_\alpha(T)$) whenever $D \subset Z$ is closed.

We denote by LB_α (resp. UB_α) the family of all set-valued maps of the lower (resp. upper) Borel class α. Taking into account (2.1) it is clear that

(2.5) $\Phi \in LB_0$ (resp. $\Phi \in UB_0$) if and only if Φ is lower (resp. upper) semicontinuous.

As regards the mutual relationship between the lower and the upper Borel classes the following is known (see [4]).

(2.6) Let $(Z, \mathcal{T}(Z))$ be a perfect topological space.

(i) If $\Phi\colon T \rightsquigarrow Z$ is closed valued and $\Phi \in UB_\alpha$, then $\Phi \in LB_{\alpha+1}$.

(ii) If moreover $(Z, \mathcal{T}(Z))$ is normal, Φ is compact valued and $\Phi \in LB_\alpha$, then $\Phi \in UB_{\alpha+1}$.

3. Main results

Now we will consider set-valued maps of two variables.

Let X and Y be nonempty sets and $(x_0, y_0) \in X \times Y$. Let $F\colon X \times Y \rightsquigarrow Z$ be a set-valued map. The set-valued map $F_{x_0}\colon Y \rightsquigarrow Z$ defined by $F_{x_0}(y) = F(x_0, y)$ we will call x_0-section of F and the set-valued map $F^{y_0}\colon X \rightsquigarrow Z$ defined by $F^{y_0}(x) = F(x, y_{y_0})$, y_0-section of F.

Similarly, for a set $E \subset X \times Y$ the sets $E_{x_0} = \{y \in Y : (x_0, y) \in E\}$ and $E^{y_0} = \{x \in X : (x, y_0) \in E\}$ we will call x_0-section of E and y_0-section of E, respectively.

Theorem 3.1. *Let $(X, \mathcal{T}(X))$ be a perfect topological space, (Y, d) be a separable metric space, $(Z, \mathcal{T}(Z))$ be a perfectly normal topological space and $F\colon X \times Y \rightsquigarrow Z$. If all y-sections of F are of the upper class α and all x-sections of F are lower semicontinuous and upper quasicontinuous, then F is of the lower class $\alpha + 1$.*

Dôkaz. Let $D \subset Z$ be an arbitrary closed set. Taking into account (2.4) it is enough to prove that

$$F^+(D) \in \prod_{\alpha+1} (X \times Y). \tag{3.1}$$

Indeed, for each $G \in \mathcal{T}(Z)$ we will have $F^-(G) = X \setminus F^+(Z \setminus G) \in \sum_{\alpha+1}(X \times Y)$, i.e. $F \in LB_{\alpha+1}(X \times Y)$.

By the perfect normality of Z there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of open subsets of Z such that

$$D = \bigcap_{n\in\mathbb{N}} G_n = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(G_n) \quad \text{and} \quad \mathrm{Cl}(G_{n+1}) \subset G_n \text{ for each } n \in \mathbb{N}. \tag{3.2}$$

Let $S = \{y_k : k \in \mathbb{N}\}$ be a dense subset of Y. In order to get (3.1) we show that

$$F^+(D) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x \in X : F(x, y_k) \subset G_n\} \times B(y_k, 2^{-n})). \tag{3.3}$$

Suppose that $(u, v) \in F^+(D)$. Then $F(u, v) \subset D$ and then, by (3.2), $F(u, v) \subset G_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since F_u is upper quasicontinuous at v, there is a nonempty open set $U \subset B(v, 2^{-n})$ such that $F(u, y) \subset G_n$ for each $y \in U$. Let $k \in \mathbb{N}$ be such that $y_k \in U$. Then $F(u, y_k) \subset G_n$ and $v \in B(y_k, 2^{-n})$, and then $(u, v) \in (F^{y_k})^+(G_n) \times B(y_k, 2^{-n})$. Thus

$$F^+(D) \subset \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x \in X : F(x, y_k) \subset G_n\} \times B(y_k, 2^{-n})). \tag{3.4}$$

Now we show the reverse inclusion. Let us suppose, to get a contradiction, that

$$(u, v) \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x \in X : F(x, y_k) \subset G_n\} \times B(y_k, 2^{-n})) \setminus F^+(D).$$

Then, by (3.2),

$$(u, v) \in (X \times Y) \setminus F^+(D) = F^-(Z \setminus \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(G_n)) = \bigcup_{n\in\mathbb{N}} F^-(Z \setminus \mathrm{Cl}(G_n)).$$

Therefore exists $m \in \mathbb{N}$ such that

$$F(u, v) \cap (Z \setminus \mathrm{Cl}(G_m)) \neq \emptyset.$$

Since F_u is lower semicontinuos at v, there is an open neighbourhood W of v such that

$$F(u, y) \cap (Z \setminus \mathrm{Cl}(G_m)) \neq \emptyset \quad \text{for all}\, y \in W. \tag{3.5}$$

Let $M \in \mathbb{N}$ be such that $B(v, 2^{-M}) \subset W$. Let $l \in \mathbb{N}$ be such that $l > M$ and $l > m$. Moreover

$$(u, v) \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x \in X : F(x, y_k) \subset G_n\} \times B(y_k, 2^{-n})).$$

Hence, there must exist $k \in \mathbb{N}$ such that

$$(u, v) \in \{x \in X : F(x, y_k) \subset G_l\} \times B(y_k, 2^{-l}).$$

Thus $y_k \in B(v, 2^{-l}) \subset W$ and $F(u, y_k) \subset G_l \subset G_m$, a contradiction to (3.5). Therefore

$$\bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x \in X : F(x, y_k) \subset G_n\} \times B(y_k, 2^{-n})) \subset F^+(D).$$

This together with (3.4) proves (3.3).

Note that

$$\{x \in X : F(x, y_k) \subset G_n\} = (F^{y_k})^+(G_n) \in \sum_\alpha(X),$$

because, by assumption, $F^{y_k} \in UB_\alpha(X)$. Thus

$$\{x \in X : F(x, y_k) \subset G_n\} \times B(y_k, 2^{-n}) \in \sum_\alpha(X \times Y),$$

what is proving (3.1) and finishes the proof. □

Now let us consider the following example (cf. [12, Example 4]).

Example 3.1. Let $A \notin \mathcal{B}(\mathbb{R})$ and $B = \mathbb{R} \setminus A$ and let $F\colon \mathbb{R} \times \mathbb{R} \rightsquigarrow \mathbb{R}$ be defined as follows:

$$F(x, y) = \begin{cases} [-2, 2] & \text{if } x \neq y, \\ [-1, 0] & \text{if } x = y \in A, \\ [0, 1] & \text{if } x = y \in B. \end{cases}$$

Then the sections F_x and F^y are lower semicontinuous for each $x, y \in \mathbb{R}$. Thus, see (2.5), $F^y \in LB_0$ and, by (2.6) (ii), $F^y \in UB_1$. But $F \notin LB_2$, since

$$\begin{aligned} F^-((-2, 0)) &= \{(x, y) : F(x, y) \cap (-2, 0) \neq \emptyset\} \\ &= \mathbb{R}^2 \setminus \{((x, x) : x \in B\} \notin \mathcal{B}(\mathbb{R}^2). \end{aligned}$$

More precisely, F is neither of any lower nor of any upper class α, since

$$F^+((-2, 1)) = \{(x, y) : F(x, y) \subset (-2, 1)\} = \{(x, x) : x \in A\} \notin \mathcal{B}(\mathbb{R}^2).$$

This example shows that if in Theorem 3.1 we suppose lower semicontinuity instead of lower semicontinuity and upper quasicontinuity, then the result is not true.

The next theorem is dual to Theorem 3.1. There is however no duality between the classes LB_α and UB_α. So, there should be carried out an independent proof.

THEOREM 3.2. *Let $(X, \mathcal{T}(X))$ be a perfect topological space, (Y, d) be a separable metric space and $(Z, \mathcal{T}(Z))$ be a perfectly normal topological space. If $F\colon X \times Y \rightsquigarrow Z$ is countably compact valued, all its y-sections are of the lower class α and all its x-sections are lower quasicontinuous and upper semicontinuous, then F is of the upper class $\alpha + 1$.*

DÔKAZ. Let $D \subset Z$ be an arbitrary closed set. Similarly as in the proof of Theorem 3.1, let $(G_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of Z such that

$$D = \bigcap_{n\in\mathbb{N}} G_n = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(G_n) \quad \text{and} \quad \mathrm{Cl}(G_{n+1}) \subset G_n \text{ for each } n \in \mathbb{N}. \tag{3.6}$$

Also let $S = \{y_k : k \in \mathbb{N}\}$ be a dense subset of Y. For the proof of this theorem it is enough to show that

$$F^-(D) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x, y_k) \cap G_n \neq \emptyset\} \times B(y_k, 2^{-n})). \tag{3.7}$$

Indeed, by the assumption, $F^{y_k} \in LB_\alpha(X)$ for each $k \in \mathbb{N}$. So, $(F^{y_k})^-(G_n) \in \sum_\alpha(X)$. Hence it follows that

$$\{x : F(x, y_k) \cap G_n \neq \emptyset\} \times B(y_k, 2^{-n}) \in \sum_\alpha(X \times Y),$$

and hence $F^-(D) \in \prod_{\alpha+1}(X \times Y)$. Thus for each $G \in \mathcal{T}(Z)$

$$F^+(G) = (X \times Y) \setminus F^-(Z \setminus G) \in \sum_{\alpha+1}(X \times Y),$$

i.e. $F \in UB_{\alpha+1}(X \times Y)$.

So, we need to prove (3.7). Suppose that $(u, v) \in F^-(D)$. Then $F(u, v) \cap D \neq \emptyset$ and then, by (3.6),

$$F(u, v) \cap G_n \neq \emptyset \quad \text{for each } n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$. By lower quasicontinuity of F_u at the point v, there is a nonempty open set $U \subset B(v, 2^{-n})$ such that

$$F(u, y) \cap G_n \neq \emptyset \quad \text{for all } y \in U.$$

There is $k \in \mathbb{N}$ such that $y_k \in S \cap U$ and $v \in B(y_k, 2^{-n})$, i.e.

$$(u,v) \in \{x : F(x,y_k) \cap G_n \neq \emptyset\} \times B(y_k, 2^{-n}).$$

In the process

$$F^-(D) \subset \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x,y_k) \cap G_n \neq \emptyset\} \times B(y_k, 2^{-n})). \tag{3.8}$$

Now we show the inverse inclusion. Suppose, opposite, that

$$(u,v) \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x,y_k) \cap G_n \neq \emptyset\} \times B(y_k, 2^{-n})) \setminus F^-(D).$$

$(u,v) \notin F^-(D)$, thus $F(u,v) \subset Z \setminus D$. Taking into consideration (3.6) it is clear that

$$F(u,v) \subset \bigcup_{n\in\mathbb{N}} (Z \setminus \mathrm{Cl}(G_n)).$$

By the assumption, the set $F(u,v)$ is countably compact. Moreover $\{Z \setminus \mathrm{Cl}(G_n)\}_{n\in\mathbb{N}}$ is an increasing open cover of $F(u,v)$. Thus

$$\exists\, m \in \mathbb{N} \;\; F(u,v) \subset Z \setminus \mathrm{Cl}(G_m).$$

So, by upper semicontinuity of F_u at the point v,

$$\exists\, W(v) \in \mathcal{O}(v)\, \forall\, y \in W(v) \;\; F(u,y) \subset Z \setminus \mathrm{Cl}(G_m).$$

Let $M \in \mathbb{N}$ be such that $B(v, 2^{-M}) \subset W(v)$. Let $l \in \mathbb{N}$ be such that $l > m$ and $l > M$. By the assumption there must exist $k \in \mathbb{N}$ such that $(u,v) \in \{x : F(x,y_k) \cap G_l \neq \emptyset\} \times B(y_k, 2^{-l}))$. Thus $F(u,y_k) \cap G_m \neq \emptyset$ and $y_k \in B(v, 2^{-M})$, a contradiction. Therefore

$$\bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (\{x : F(x,y_k) \cap G_n \neq \emptyset\} \times B(y_k, 2^{-n})) \subset F^-(D).$$

Hence, taking into account (3.8), we show (3.7) and finish the proof. □

Now consider the next example (cf. [12, Example 2]).

Example 3.2. Let $A \notin \mathcal{B}(\mathbb{R})$ and $B = \mathbb{R} \setminus A$ and let $F \colon \mathbb{R} \times \mathbb{R} \rightsquigarrow \mathbb{R}$ be defined as follows:

$$F(x,y) = \begin{cases} [0,1] & \text{if } x \neq y, \\ [0,2] & \text{if } x = y \in A, \\ [-2,1] & \text{if } x = y \in B. \end{cases}$$

Then the sections F_x and F^y are upper semicontinuous for each $x, y \in \mathbb{R}$. So, see (2.5), $F^y \in UB_0$ and, by (2.6) (i), $F^y \in LB_1$ for each $y \in Y$. But $F \notin UB_2$, because

$$F^+((-1,3)) = \{(x,y) : F(x,y) \subset (-1,3)\} = \mathbb{R}^2 \setminus \{((x,x) : x \in B\} \notin \mathcal{B}(\mathbb{R}^2).$$

More precisely, F is neither of any lower nor of any upper class α, since

$$F^-((-1,0)) = \{(x,y) : F(x,y) \cap (-1,0) \neq \emptyset\} = \{(x,x) : x \in B\} \notin \mathcal{B}(\mathbb{R}^2).$$

It is known that if in Theorem 3.2 we suppose that $F^y \in LB_0$ for any $y \in Y$ and F_x is only upper semicontinuous for any $x \in X$, then $F \in UB_1$ (see [11, Theorem 1]). Example 3.2 shows that Theorem 3.2 does not work if we replace assumptions on F_x (lower quasicontinuity and upper semicontinuity), by upper semicontinuity of F_x.

We also have the following result.

THEOREM 3.3. *Let $(X, \mathcal{T}(X))$ be a perfectly normal topological space and (Y,d), (Z,ρ) be separable metric spaces. Suppose that $F\colon X \times Y \rightsquigarrow Z$ is closed valued such that all its y-sections are of the lower class α and all x-sections are lower semicontinuous and upper quasicontinuous. Then F is of the lower class $\alpha + 2$.*

DÔKAZ. Let $z \in Z$ be fixed. Let us define a function $g_z\colon X \times Y \to \mathbb{R}$ by

$$g_z(x,y) = \rho(z, F(x,y)).$$

For the proof of the theorem it is enough to show that the function g_z is of the upper Young class $\alpha + 2$, i.e.

$$\forall\, r \in \mathbb{R}_+ \; g_z^{-1}((-\infty, r)) \in \sum_{\alpha+2} (X \times Y). \tag{3.9}$$

Indeed, for given $r \in \mathbb{R}$ we have

$$\begin{aligned} g_z^{-1}((-\infty, r)) &= \{(x,y) \in X \times Y : \rho(z, F(x,y)) < r\} \\ &= \{(x,y) \in X \times Y : F(x,y) \cap B(z,r) \neq \emptyset\} = F^-(B(z,r)). \end{aligned}$$

Then, in accordance with (3.9),

$$F^-(B(z,r)) \in \sum_{\alpha+2}(X \times Y).$$

Thus, by separability of Z, for an open set $G \subset Z$ we have

$$F^-(G) = F^-(\bigcup_{i\in\mathbb{N}} B(z_i,r_i)) = \bigcup_{i\in\mathbb{N}} F^-(B(z_i,r_i)) \in \sum_{\alpha+2}(X \times Y),$$

where $\{z_i : i \in \mathbb{N}\}$ is a countable dense set in G and $\{r_i : i \in \mathbb{N}\}$ is a sequence of positive real numbers such that $B(z_i, r_i) \subset G$ for each $i \in \mathbb{N}$. Thus $F \in LB_{\alpha+2}(X \times Y)$ and the assertion follows.

So, it remains to prove (3.9). Let $r \in \mathbb{R}_+$ and $(x, y) \in X \times Y$. Since F^y is of the lower class α, we conclude that

$$(g_z^y)^{-1}((-\infty, r)) = (F^y)^-(B(z,r)) \in \sum_{\alpha}(X).$$

Hence g_z^y is of the upper Young class α.

Moreover, F_x is lower semicontinuous. So, by (2.2) (i),

$$(g_z)_x \quad \text{is upper semicontinuous.} \tag{3.10}$$

F_x is upper quasicontinuous, so by (2.3) (ii),

$$(g_z)_x \quad \text{is lower quasicontinuous.} \tag{3.11}$$

Let $H\colon X \rightsquigarrow Y \times \mathbb{R}$ be defined as follows:

$$H(x) = \{(y,r) \in Y \times \mathbb{R} : g_z(x,y) \geq r\}.$$

Note that, by (3.10),

$$\forall\, x \in X \ H(x) \quad \text{is a closed subset of } Y \times \mathbb{R}. \tag{3.12}$$

Let $S = \{s_n\}_{n\in\mathbb{N}}$ be a dense subset of Y and $(r_m)_{m\in\mathbb{N}}$ be the sequence of all rational numbers. Define a sequence of functions $f_{nm}\colon X \to Y \times \mathbb{R}$ by

$$f_{nm}(x) = (s_n, \min\{r_m, g_z(x,s_n)\}).$$

Then $f_{nm}(x) \in H(x)$ for each $x \in X$ and $m, n \in \mathbb{N}$. Thus $\{f_{nm}(x) : n, m \in \mathbb{N}\} \subset H(x)$ for $x \in X$. So, by (3.12), for each $x \in X$ we have the following inclusion

$$\mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\}) \subset H(x). \tag{3.13}$$

Suppose that $(y, r) \in H(x)$, i.e. $g_z(x, y) \geq r$. Let $O_y \in \mathcal{O}(y)$ and $O_r \in \mathcal{O}(r)$. There is $m \in \mathbb{N}$ such that $r_m < r$ and $r_m \in O_r$. By (3.11) there is a nonempty open set G, $G \subset O_y$ such that $g_z(x, v) > r_m$ for every $v \in G$. Let $n \in \mathbb{N}$ be such that $s_n \in G$. Then $f_{nm}(x) \in O_y \times O_r$.

This way we demonstrated that

$$(y, r) \in \mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\}).$$

Thus, $H(x) \subset \mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\})$. Taking into account of (3.13) we have

$$H(x) = \mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\}).$$

Note that

$$f_{nm} \quad \text{is of the Borel class } \alpha + 1 \text{ for each } n, m \in \mathbb{N}. \tag{3.14}$$

Now let $W \subset Y \times \mathbb{R}$ be an open set. Then

$$\begin{aligned} H^-(W) &= \{x \in X : H(x) \cap W \neq \emptyset\} \\ &= \{x \in X : \mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\}) \cap W \neq \emptyset\} \\ &= \{x \in X : \{f_{nm}(x) : n, m \in \mathbb{N}\} \cap W \neq \emptyset\} \\ &= \bigcup_{n,m \in \mathbb{N}} \{x \in X : f_{nm}(x) \in W\} = \bigcup_{n,m \in \mathbb{N}} f^{-1}_{n,m}(W), \end{aligned}$$

and then, by (3.14), $H^-(W) \in \sum_{\alpha+1}(X)$. Thus

$$H \in LB_{\alpha+1}(X). \tag{3.15}$$

Now we consider the graph of the set-valued map H, i.e. the set

$$Gr(H) = \{(x, y, r) \in X \times Y \times \mathbb{R} : (y, r) \in H(x)\}.$$

Let $\{G_i\}_{i \in \mathbb{N}}$ be a countable base of the space $Y \times \mathbb{R}$. Note that

$$(y, r) \notin H(x) \Rightarrow \exists\, n \in \mathbb{N}\, (y, r) \in G_n \wedge H(x) \cap G_n = \emptyset.$$

Therefore

$$
\begin{aligned}
(X \times Y \times \mathbb{R}) \setminus Gr(H) &= \bigcup_{n\in\mathbb{N}} (\{x \in X : H(x) \cap G_n = \emptyset\} \times G_n) \\
&= \bigcup_{n\in\mathbb{N}} (\{x \in X : H(x) \subset (Y \times \mathbb{R}) \setminus G_n)\} \times G_n) \\
&= \bigcup_{n\in\mathbb{N}} (H^+((Y \times \mathbb{R}) \setminus G_n)) \times G_n,
\end{aligned}
$$

and we have

$$
(3.16) \qquad (X \times Y \times \mathbb{R}) \setminus Gr(H) = \bigcup_{n\in\mathbb{N}} (H^+((Y \times \mathbb{R}) \setminus G_n)) \times G_n.
$$

Hence, by (3.15),

$$
H^+((Y \times \mathbb{R}) \setminus G_n) = X \setminus H^-(G_n) \in \prod_{\alpha+1}(X) \subset \sum_{\alpha+2}(X),
$$

and hence that

$$
H^+((Y \times \mathbb{R}) \setminus G_n) \times G_n \in \sum_{\alpha+2}(X \times Y \times \mathbb{R}).
$$

Finally, by (3.16),

$$
(X \times Y \times \mathbb{R}) \setminus Gr(H) \in \sum_{\alpha+2}(X \times Y \times \mathbb{R}).
$$

Thus $Gr(H) \in \prod_{\alpha+2}(X \times Y \times \mathbb{R})$ and thereby

$$
(3.17) \qquad (Gr(H))^r = \{(x, y) \in X \times Y : (x, y, r) \in Gr(H)\} \in \prod_{\alpha+2}(X \times Y).
$$

On the other hand

$$
\begin{aligned}
(Gr(H))^r &= \{(x, y) \in X \times Y : g_z(x, y) \geq r\} \\
&= (X \times Y) \setminus \{(x, y) \in X \times Y : g_z(x, y) < r\} \\
&= (X \times Y) \setminus g_z^{-1}((-\infty, r)).
\end{aligned}
$$

Taking into account (3.17), we have

$$g_z^{-1}((-\infty, r)) \in \sum_{\alpha+2}(X \times Y),$$

which proves (3.9) and finishes the proof of theorem. □

4. Final Remarks

In various problems of mathematics one encounters measurability of set-valued maps of two variables. Particular emphasis is led on set-valued maps having the Scorca–Dragoni property, it means, loosely speaking, set-valued maps which are continuous "up to small sets". The importance of such maps for differential inclusions (i.e. differential equations with set-valued right-hand sides) is the same as in the single-valued case for differential equations. However, if we replace the continuity by a weaker semi-continuity (i.e. by lower semicontinuity or upper semicontinuity), many new features occur. The most complete presentation of set-valued maps having the Scorza–Dragoni properties is contained in thesis [14]. In that paper some relations between measurability of set-valued maps of two variables and having the lower Scorza–Dragoni or upper Scorza–Dragoni property are established.

The theory of $\mathcal{B}$-measurability of set-valued maps of two variables which has been outlined in our paper gives some possibility of appylying this theory to research of the Scorza–Dragoni property of set-valued maps.

Literatúra

[1] R. Brisac, *Les classes de Baire des fonctions multiformes*, C. R. Acad. Sci. Paris **224** (1947), 257–258.

[2] J. Ewert, *Multivalued Mappings and Bitopological Spaces* (Polish), Pomeranian University in Słupsk, Słupsk, 1985.

[3] J. Ewert and T. Lipski, *Lower and upper quasicontinuous functions*, Demonstratio Math. **16** (1983), no. 1, 85–93.

[4] K.M. Garg, *On the classification of set-valued functions*, Real Anal. Exchange **9** (1983/84), no. 1, 86–93.

[5] R.W. Hansell, *Hereditarily additive families in descriptive set theory and Borel measurable multimaps*, Trans. Amer. Math. Soc. **278** (1983), no. 2, 725–749.

[6] S. Hu and N.S. Papageorgiou, *Handbook of Multivalued Analysis*, vol. I, Kluwer Academic Publishers, Dordrecht-Boston-London, 1997.

[7] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math. **19** (1932), 184–197.

[8] K. Kuratowski, *Sur la théorie des fonctions dans les espaces métriques*, Fund. Math. **17** (1931), 275–282.

[9] K. Kuratowski, *On set-valued B-measurable mappings and a theorem of Hausdorff*, in: G. Asser, J. Flachsmeyer, and W. Rinow (eds.), *Theory of Sets and Topology (in Honour of Felix Hausdorff, 1868-1942)*, VEB Deutsh. Verlag Wissench., Berlin, 1972, pp. 355–362.

[10] K. Kuratowski, *Some remarks on the relation of classical set-valued mappings to the Baire classification*, Colloq. Math. **42** (1979), 273–277.

[11] G. Kwiecińska, *On the Borel class of multivalued functions of two variables*, Topology Proc. **25** (2000), 601–613.

[12] G. Kwiecińska, *B-measurability of multifunctions of two variables*, Real Analysis Exchange, Summer Symposium 2011, 36–41.

[13] T. Neubrunn, *Quasi-continuity*, Real Anal. Exchange **14** (1988), no. 2, 259–306.

[14] W. Zygmunt, *The Scorza–Dragoni Property* (Polish), Thesis, M. Curie-Skłodowska University, Lublin, 1990.

Ľubica Holá
Academy of Sciences
Institute of Mathematics
Štefánikova 49
81473 Bratislava
Slovakia
e-mail: hola@mat.savba.sk

Grażyna Kwiecińska
University of Gdańsk
Institute of Mathematics
Wita Stwosza
Gdańsk
Poland
e-mail: gkk@mat.ug.edu.pl