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A UNIQUE COMMON FIXED POINT FOR AN INFINITY OF SET-VALUED MAPS

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Abstract. The main purpose of this paper is to establish some common fixed point theorems for single and set-valued maps in complete metric spaces, under contractive conditions by using minimal type commutativity and without continuity. These theorems generalize, extend and improve the result due to Elamrani and Mehdaoui ([2]) and others. Also, common fixed point theorems in metric spaces under strict contractive conditions are given.

1. Introduction

The theory of common fixed point theorems of single and set-valued maps is very rich. It provides some techniques for solving numerous problems in mathematical science and engineering. As in the single-valued setting, many authors have studied the existence of fixed and common fixed points for single and set-valued maps for contractive and strictly contractive maps in metric as well as in compact metric spaces.

Our work here establishes common fixed point theorems for single and set-valued maps under contractive conditions. These theorems use minimal type commutativity with no continuity requirements. Our theorems generalize some results, especially the theorem due to Elamrani and Mehdaoui ([2]). Also we give some results in metric spaces under strictly contractive conditions which include neither continuity nor compactness.

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2. Preliminaries

Throughout this paper, $(\mathcal{X}, d)$ denotes a metric space and $B(\mathcal{X})$ is the set of all nonempty bounded subsets of $\mathcal{X}$. As in [9] and [5], we define the functions $\delta(A, B)$ and $D(A, B)$ as follows:

$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\},$$

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$

for all A, B in $B(\mathcal{X})$. If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. Also, if $B = \{b\}$, it yields $\delta(A, B) = d(a, b)$.

The definition of the function $\delta(A, B)$ yields the following:

$$\delta(A,B) = \delta(B,A),$$

$$\delta(A,B) \le \delta(A,C) + \delta(C,B),$$

$$\delta(A,B) = 0 \text{ if and only if } A = B = \{a\},$$

$$\delta(A,A) = \operatorname{diam} A,$$

for all A, B, C in $B(\mathcal{X})$.

A subset A of $\mathcal{X}$ is the limit of a sequence $\{A_n\}$ of non-empty subsets of $\mathcal{X}$ if each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n = 1, 2, \ldots$, and if for arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$, where A_ε is the union of all open spheres with centers in A and radius ε (see [9]).

Lemma 2.1 ([9]). *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of $(\mathcal{X}, d)$ which converge to the bounded sets A and B respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Let F be a map of $\mathcal{X}$ into $B(\mathcal{X})$. F is continuous at the point x in $\mathcal{X}$ if for any sequence $\{x_n\}$ in $\mathcal{X}$ converging to x, the sequence $\{Fx_n\}$ in $B(\mathcal{X})$ converges to Fx in $B(\mathcal{X})$ ([9]).

Definition 2.2 ([10]). Maps $\mathcal{T} \colon \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{K} \colon \mathcal{X} \to \mathcal{X}$ are said to be *weakly commuting on $\mathcal{X}$* if for any $x \in \mathcal{X}$:

$$\delta(\mathcal{K}\mathcal{T}x, \mathcal{T}\mathcal{K}x) \le \max\{\delta(\mathcal{K}x, \mathcal{T}x), \operatorname{diam} \mathcal{K}\mathcal{T}x\}.$$

If $\mathcal{T}$ is a single-valued map, then $\operatorname{diam} \mathcal{K}\mathcal{T}x = 0$ for all $x \in \mathcal{X}$ because the set $\mathcal{K}\mathcal{T}x$ contains a single point and the above inequality reduces to the condition given by Sessa (see [8]) as follows:

$$d(\mathcal{K}\mathcal{T}x, \mathcal{T}\mathcal{K}x) \leq d(\mathcal{T}x, \mathcal{K}x)$$

for all $x \in \mathcal{X}$.

Clearly, two commuting maps $\mathcal{T}$ and $\mathcal{K}$ ($\mathcal{T}\mathcal{K}x = \mathcal{K}\mathcal{T}x, x \in \mathcal{X}$) are weakly commuting but the converse is not necessarily true.

In 1986, Jungck ([3]) introduced extension of weakly commuting maps for single-valued maps by proposing the following definition.

Definition 2.3 ([3]). Two single-valued maps f and g of a metric space $(\mathcal{X}, d)$ into itself are *compatible* if and only if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

It is well known that weakly commuting single-valued maps are compatible but the converse need not be true, as is shown in [3].

In 1993, Jungck and Rhoades ([4]) extended the above definition to set-valued maps, as follows:

Definition 2.4 ([4]). Maps $\mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ are *δ-compatible* if

$$\lim_{n\to\infty} \delta(\mathcal{T}\mathcal{K}x_n, \mathcal{K}\mathcal{T}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{T}x_n \to \{t\}$ and $\mathcal{K}x_n \to t$ for some $t \in \mathcal{X}$ and $\mathcal{K}\mathcal{T}x_n \in B(\mathcal{X})$.

Motivated by the above definition, the same authors ([5]) gave this generalization:

Definition 2.5 ([5]). Maps $\mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ are *weakly compatible* if and only if $\mathcal{T}x = \{\mathcal{K}x\}$ implies that $\mathcal{T}\mathcal{K}x = \mathcal{K}\mathcal{T}x$.

Before observing that δ-compatible maps are weakly compatible, we must include the following definitions.

Definition 2.6. Let $\mathcal{T}$ be a map of $\mathcal{X}$ into $B(\mathcal{X})$. We define

$$\mathcal{T}(\mathcal{X}) = \{\mathcal{T}(x) : x \in \mathcal{X}\}.$$

Definition 2.7. Let $\mathcal{T}$ be a map of $\mathcal{X}$ into $B(\mathcal{X})$. We define

$$\cup\mathcal{T}(\mathcal{X}) = \underset{x\in\mathcal{X}}{\cup}\mathcal{T}(x).$$

Now, it can be seen that two weakly commuting set-valued maps are δ-compatible, but in general the converse is false.

Also, δ-compatible maps are weakly compatible but the converse is not true. Examples supporting this can be found in [5]. To confirm this fact, let us consider the following example.

Example 2.8. Let $\mathcal{X} = [0, 2]$ with the usual metric d. Define

$$\mathcal{K}x = \begin{cases} 1 & \text{if } x \in [0, 1), \\ 2 - x & \text{if } x \in [1, 2], \end{cases} \qquad \mathcal{T}x = \begin{cases} [0, 1] & \text{if } x \in [0, 1), \\ [1, x] & \text{if } x \in [1, 2]. \end{cases}$$

Obviously, $\mathcal{K}$ and $\mathcal{T}$ are weakly compatible maps, since they commute at coincidence point $x = 1$. Consider the sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_n = 1 + \frac{1}{n}, n \in \mathbb{N}^*$. Then,

$$\mathcal{K}x_n = 2 - x_n \to 1 \text{ as } x_n \to 1 \quad \text{and} \quad \mathcal{T}x_n = [1, x_n] \to \{1\} \text{ as } x_n \to 1.$$

On the other hand, we have $\mathcal{K}\mathcal{T}x_n \in B(\mathcal{X})$ and

$$\delta(\mathcal{T}\mathcal{K}x_n, \mathcal{K}\mathcal{T}x_n) = \delta([0, 1], [2 - x_n, 1]) \to 1 \neq 0,$$

this tells that $\mathcal{K}$ and $\mathcal{T}$ are not δ-compatible.

In [6], Khan has established fixed point theorems for self-maps of a complete metric space by altering the distance between points by means of a continuous and strictly increasing function $\Phi\colon [0, \infty) \to [0, \infty)$ such that

$$\Phi(t) = 0 \quad \text{if and only if} \quad t = 0. \tag{2.1}$$

Following this technique, Elamrani and Mehdaoui ([2]) established a theorem of a common fixed point for compatible and weakly compatible single and set-valued maps in complete metric spaces.

The objective here is to generalize, improve and extend the result of [2] by using minimal type commutativity and without assumption of continuity.

3. Main results

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{J}, \mathcal{K}$ be single-valued maps from $\mathcal{X}$ into itself. Let $\mathcal{S}, \mathcal{T} \colon \mathcal{X} \to B(\mathcal{X})$ be set-valued maps such that*

$$\cup\mathcal{T}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X}) \quad \textit{and} \quad \cup\,\mathcal{S}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X}).$$

Let Φ be an increasing and continuous function of $[0, \infty)$ into itself satisfying property (2.1) *and inequality*

$$\begin{aligned}(3.1)\quad \Phi(\delta(\mathcal{T}x, \mathcal{S}y)) \le {} & a(d(\mathcal{K}x, \mathcal{J}y))\Phi(d(\mathcal{K}x, \mathcal{J}y)) \\ & + b(d(\mathcal{K}x, \mathcal{J}y))\,[\Phi(\delta(\mathcal{K}x, \mathcal{T}x)) + \Phi(\delta(\mathcal{J}y, \mathcal{S}y))] \\ & + c(d(\mathcal{K}x, \mathcal{J}y)) \min\{\Phi(D(\mathcal{K}x, \mathcal{S}y)), \Phi(D(\mathcal{J}y, \mathcal{T}x))\}\end{aligned}$$

for all x, y in $\mathcal{X}$, where $a, b, c \colon [0, \infty) \to [0, 1)$ are continuous increasing functions satisfying condition

$$(3.2)\qquad a(t) + 2b(t) + c(t) < 1, \quad t > 0.$$

If the pairs of maps $\{\mathcal{T}, \mathcal{K}\}$ and $\{\mathcal{J}, \mathcal{S}\}$ are weakly compatible and either

$\mathcal{T}(\mathcal{X})$ or $\mathcal{S}(\mathcal{X})$ (resp. $\mathcal{J}(\mathcal{X})$ or $\mathcal{K}(\mathcal{X})$) is closed,

then $\mathcal{J}, \mathcal{K}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point t in $\mathcal{X}$, i.e.

$$\mathcal{S}t = \mathcal{T}t = \{\mathcal{J}t\} = \{\mathcal{K}t\} = \{t\}.$$

Proof. Let $x_0 \in \mathcal{X}$ be given. Since $\cup\mathcal{T}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$, then there exists a point $x_1 \in \mathcal{X}$ such that $\mathcal{J}x_1 \in \mathcal{T}x_0 = Y_1$. For this point x_1, since $\cup\mathcal{S}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$, there is another point $x_2 \in \mathcal{X}$ such that $\mathcal{K}x_2 \in \mathcal{S}x_1 = Y_2$. Continuing in this way, we can produce by induction a sequence in $\mathcal{X}$ such that

$$(3.3)\quad \mathcal{J}x_{2n+1} \in \mathcal{T}x_{2n} = Y_{2n+1}, \mathcal{K}x_{2n+2} \in \mathcal{S}x_{2n+1} = Y_{2n+2} \ \text{ for all } n \in \mathbb{N}.$$

For simplicity, we set

$$\delta_n = \delta(Y_n, Y_{n+1}), \quad n \in \mathbb{N}.$$

From inequality (3.1) it follows that

$$\begin{aligned}\Phi(\delta_{2n+1}) &= \Phi(\delta(Y_{2n+1}, Y_{2n+2})) = \Phi(\delta(\mathcal{T}x_{2n}, \mathcal{S}x_{2n+1}))\\ &\leq a(d(\mathcal{K}x_{2n}, \mathcal{J}x_{2n+1}))\Phi(d(\mathcal{K}x_{2n}, \mathcal{J}x_{2n+1}))\\ &\quad + b(d(\mathcal{K}x_{2n}, \mathcal{J}x_{2n+1}))\left[\Phi(\delta(\mathcal{K}x_{2n}, \mathcal{T}x_{2n}))\right.\\ &\quad \left.+\Phi(\delta(\mathcal{J}x_{2n+1}, \mathcal{S}x_{2n+1}))\right]\\ &\quad + c(d(\mathcal{K}x_{2n}, \mathcal{J}x_{2n+1}))\min\{\Phi(D(\mathcal{K}x_{2n}, \mathcal{S}x_{2n+1})),\\ &\quad \Phi(D(\mathcal{J}x_{2n+1}, \mathcal{T}x_{2n}))\}\end{aligned}$$

for $n \in \mathbb{N}$. Since $\mathcal{J}x_{2n+1} \in \mathcal{T}x_{2n}$ then

$$c(d(\mathcal{K}x_{2n}, \mathcal{J}x_{2n+1}))\min\{\Phi(D(\mathcal{K}x_{2n}, \mathcal{S}x_{2n+1})), \Phi(D(\mathcal{J}x_{2n+1}, \mathcal{T}x_{2n}))\} = 0,$$

which implies that

$$\Phi(\delta_{2n+1}) \leq a(\delta_{2n})\Phi(\delta_{2n}) + b(\delta_{2n})\left[\Phi(\delta_{2n}) + \Phi(\delta_{2n+1})\right],$$

so that, taking (3.2) into account,

$$\Phi(\delta_{2n+1}) \leq \frac{a(\delta_{2n}) + b(\delta_{2n})}{1 - b(\delta_{2n})}\Phi(\delta_{2n}) < \Phi(\delta_{2n}). \tag{3.4}$$

Similarly, we have

$$\Phi(\delta_{2n+2}) \leq \frac{a(\delta_{2n+1}) + b(\delta_{2n+1})}{1 - b(\delta_{2n+1})}\Phi(\delta_{2n+1}) < \Phi(\delta_{2n+1}). \tag{3.5}$$

Since Φ is increasing, $\{\delta_n\}$ is a decreasing sequence. Put $\delta = \lim_{n\to\infty} \delta_n$. Then $\delta = 0$. In fact, from (3.4) and (3.5),

$$\Phi(\delta) \leq \Phi(\delta_n) \leq \frac{a(\delta_{n-1}) + b(\delta_{n-1})}{1 - b(\delta_{n-1})}\Phi(\delta_{n-1}) \tag{3.6}$$

for all n, and letting $n \to \infty$ in (3.6) yields

$$\Phi(\delta) \leq \frac{a(\delta) + b(\delta)}{1 - b(\delta)}\Phi(\delta)$$

which, in view of (3.2), gives $\Phi(\delta) = 0$. Hence, by property (2.1), $\delta = 0$.

Let y_n be an arbitrary point in Y_n for $n \in \mathbb{N}$. We claim that $\{y_n\}$ is a Cauchy sequence. Since

$$\lim_{n\to\infty} d(y_n, y_{n+1}) \leq \lim_{n\to\infty} \delta(Y_n, Y_{n+1}) = 0,$$

it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. We proceed by contradiction. Thus, assume there exists $\varepsilon > 0$ such that for each even integer $2k, k \in \mathbb{N}$, even integers $2m(k)$ and $2n(k)$ with $2k \leq 2n(k) \leq 2m(k)$ can be found for which

$$\delta(Y_{2n(k)}, Y_{2m(k)}) > \varepsilon. \tag{3.7}$$

For each integer k, fix $2n(k)$ and let $2m(k)$ be the least even integer exceeding $2n(k)$ and satisfying (3.7). Then

$$\delta(Y_{2n(k)}, Y_{2m(k)-2}) \leq \varepsilon \quad \text{and} \quad \delta(Y_{2n(k)}, Y_{2m(k)}) > \varepsilon.$$

Hence, for each even integer $2k$ we have, by the triangle inequality,

$$\varepsilon < \delta(Y_{2n(k)}, Y_{2m(k)}) \leq \delta(Y_{2n(k)}, Y_{2m(k)-2}) + \delta_{2m(k)-2} + \delta_{2m(k)-1}.$$

Letting k tends to infinity, we obtain

$$\lim_{k\to\infty} \delta(Y_{2n(k)}, Y_{2m(k)}) = \varepsilon. \tag{3.8}$$

Moreover, by the triangle inequality we also have

$$\begin{aligned} -\delta_{2n(k)} - \delta_{2m(k)} + \delta(Y_{2n(k)}, Y_{2m(k)}) &\leq \delta(Y_{2n(k)+1}, Y_{2m(k)+1}) \\ &\leq \delta_{2n(k)} + \delta(Y_{2n(k)}, Y_{2m(k)}) + \delta_{2m(k)}, \end{aligned}$$

and therefore

$$\delta(Y_{2n(k)+1}, Y_{2m(k)+1}) \to \varepsilon \tag{3.9}$$

when $k \to \infty$. The same argument shows that

$$\begin{aligned} \delta(Y_{2n(k)+1}, Y_{2m(k)+1}) - \delta_{2n(k)} &\leq \delta(Y_{2n(k)}, Y_{2m(k)+1}) \\ &\leq \delta(Y_{2n(k)}, Y_{2m(k)}) + \delta_{2m(k)}, \end{aligned}$$

so that also

$$\delta(Y_{2n(k)}, Y_{2m(k)+1}) \to \varepsilon \text{ as } k \to \infty. \tag{3.10}$$

Also we have

$$-\delta_{2n(k)}-\delta_{2m(k)+1}-\delta_{2m(k)}+\delta(Y_{2n(k)},Y_{2m(k)}) \leq \delta(Y_{2n(k)+1},Y_{2m(k)+2}) \leq \delta(Y_{2n(k)+1},Y_{2m(k)+1})+\delta_{2m(k)+1},$$

thus,

$$\delta(Y_{2n(k)+1},Y_{2m(k)+2}) \to \varepsilon \quad \text{as } k\to\infty. \tag{3.11}$$

On the other hand, by assumption (3.1), we have

$$\begin{aligned}
&\Phi(\delta(Y_{2m(k)+2},Y_{2n(k)+1})) \\
&= \Phi(\delta(\mathcal{S}x_{2m(k)+1},\mathcal{T}x_{2n(k)})) = \Phi(\delta(\mathcal{T}x_{2n(k)},\mathcal{S}x_{2m(k)+1})) \\
&\leq a(d(\mathcal{K}x_{2n(k)},\mathcal{J}x_{2m(k)+1}))\Phi(d(\mathcal{K}x_{2n(k)},\mathcal{J}x_{2m(k)+1})) \\
&\quad + b(d(\mathcal{K}x_{2n(k)},\mathcal{J}x_{2m(k)+1}))\left[\Phi(\delta(\mathcal{K}x_{2n(k)},\mathcal{T}x_{2n(k)}))\right. \\
&\quad \left.+\Phi(\delta(\mathcal{J}x_{2m(k)+1},\mathcal{S}x_{2m(k)+1}))\right] \\
&\quad + c(d(\mathcal{K}x_{2n(k)},\mathcal{J}x_{2m(k)+1}))\min\left\{\Phi(D(\mathcal{K}x_{2n(k)},\mathcal{S}x_{2m(k)+1})),\right. \\
&\quad \left.\Phi(D(\mathcal{J}x_{2m(k)+1},\mathcal{T}x_{2n(k)}))\right\} \\
&\leq a(\delta(Y_{2m(k)},Y_{2n(k)})+\delta_{2m(k)})\Phi(\delta(Y_{2m(k)},Y_{2n(k)})+\delta_{2m(k)}) \\
&\quad + b(\delta(Y_{2m(k)},Y_{2n(k)})+\delta_{2m(k)})\left[\Phi(\delta_{2n(k)})+\Phi(\delta_{2m(k)+1})\right] \\
&\quad + c(\delta(Y_{2m(k)},Y_{2n(k)})+\delta_{2m(k)})\min\left\{\Phi(\delta(Y_{2m(k)},Y_{2n(k)}))+\delta_{2m(k)}\right. \\
&\quad \left.+\delta_{2m(k)+1},\Phi(\delta(Y_{2m(k)+1},Y_{2n(k)+1}))\right\}.
\end{aligned} \tag{3.12}$$

Thus, letting $k\to\infty$ in (3.12), from (2.1), (3.2), (3.8), (3.9), (3.10) and (3.11) we obtain

$$\Phi(\varepsilon) \leq [a(\varepsilon)+c(\varepsilon)]\Phi(\varepsilon) < \Phi(\varepsilon),$$

which is a contradiction. This proves our claim.

Since $\mathcal{X}$ is complete, the sequence $\{y_n\}$ converges in $\mathcal{X}$. Hence, the sequences $\{\mathcal{K}x_{2n}\},\{\mathcal{J}x_{2n+1}\}$ constructed in (3.3) converge to one and the same $t\in\mathcal{X}$. Furthermore, the sequences of sets $\{\mathcal{T}x_{2n}\}$ and $\{\mathcal{S}x_{2n+1}\}$ converge to the singleton $\{t\}$. Since $\{\mathcal{T}x_{2n}\}\subseteq\mathcal{T}(\mathcal{X})$ and $\mathcal{T}(\mathcal{X})$ is closed we have that $\{t\}\in\mathcal{T}(\mathcal{X})$. Consequently, $t\in\cup\mathcal{T}(\mathcal{X})\subseteq\mathcal{J}(\mathcal{X})$.

It then follows that, there exists an element $u \in \mathcal{X}$ such that $\mathcal{J}u = t$. Using inequality (3.1), we obtain

$$\begin{aligned}\Phi(\delta(\mathcal{T}x_{2n}, \mathcal{S}u)) \leq\ & a(d(\mathcal{K}x_{2n}, \mathcal{J}u))\Phi(d(\mathcal{K}x_{2n}, \mathcal{J}u)) \\ & + b(d(\mathcal{K}x_{2n}, \mathcal{J}u))\left[\Phi(\delta(\mathcal{K}x_{2n}, \mathcal{T}x_{2n})) + \Phi(\delta(\mathcal{J}u, \mathcal{S}u))\right] \\ & + c(d(\mathcal{K}x_{2n}, \mathcal{J}u)) \min\left\{\Phi(D(\mathcal{K}x_{2n}, \mathcal{S}u)), \Phi(D(\mathcal{J}u, \mathcal{T}x_{2n}))\right\}.\end{aligned}$$

If we had $\mathcal{S}u \neq \{t\}$, then by letting n tends to infinity in the above inequality, using Lemma 2.1 and conditions (2.1) and (3.2), we would get

$$\begin{aligned}\Phi(\delta(t, \mathcal{S}u)) \leq\ & a(d(t,t))\Phi(d(t,t)) \\ & + b(d(t,t))\left[\Phi(\delta(t,t)) + \Phi(\delta(t, \mathcal{S}u))\right] \\ & + c(d(t,t)) \min\left\{\Phi(D(t, \mathcal{S}u)), \Phi(D(t,t))\right\} \\ =\ & b(0)\Phi(\delta(t, \mathcal{S}u)) < \Phi(\delta(t, \mathcal{S}u)),\end{aligned}$$

a contradiction. Thus, $\mathcal{S}u = \{t\} = \{\mathcal{J}u\}$. But the maps $\mathcal{S}$ and $\mathcal{J}$ are weakly compatible, then $\mathcal{S}\mathcal{J}u = \mathcal{J}\mathcal{S}u$, i.e. $\mathcal{S}t = \{\mathcal{J}t\}$. We claim that t is a common fixed point of $\mathcal{S}$ and $\mathcal{J}$. Suppose not. Then, by estimation (3.1), we get

$$\begin{aligned}\Phi(\delta(\mathcal{T}x_{2n}, \mathcal{S}t)) \leq\ & a(d(\mathcal{K}x_{2n}, \mathcal{J}t))\Phi(d(\mathcal{K}x_{2n}, \mathcal{J}t)) \\ & + b(d(\mathcal{K}x_{2n}, \mathcal{J}t))\left[\Phi(\delta(\mathcal{K}x_{2n}, \mathcal{T}x_{2n})) + \Phi(\delta(\mathcal{J}t, \mathcal{S}t))\right] \\ & + c(d(\mathcal{K}x_{2n}, \mathcal{J}t)) \min\left\{\Phi(D(\mathcal{K}x_{2n}, \mathcal{S}t)), \Phi(D(\mathcal{J}t, \mathcal{T}x_{2n}))\right\}.\end{aligned}$$

Therefore, at infinity, by using Lemma 2.1 and properties (2.1) and (3.2), we have

$$\begin{aligned}\Phi(\delta(t, \mathcal{S}t)) \leq\ & a(d(t, \mathcal{S}t))\Phi(d(t, \mathcal{S}t)) \\ & + b(d(t, \mathcal{S}t))\left[\Phi(\delta(t,t)) + \Phi(\delta(\mathcal{S}t, \mathcal{S}t))\right] \\ & + c(d(t, \mathcal{S}t)) \min\left\{\Phi(D(t, \mathcal{S}t)), \Phi(D(\mathcal{S}t, t))\right\} \\ =\ & a(d(t, \mathcal{S}t))\Phi(d(t, \mathcal{S}t)) + c(d(t, \mathcal{S}t))\Phi(D(t, \mathcal{S}t)) \\ \leq\ & \left[a(d(t, \mathcal{S}t)) + c(d(t, \mathcal{S}t))\right]\Phi(\delta(t, \mathcal{S}t)) \\ <\ & \Phi(\delta(t, \mathcal{S}t)).\end{aligned}$$

This contradiction implies that $\mathcal{S}t = \{t\}$. Hence $\mathcal{S}t = \{\mathcal{J}t\} = \{t\}$. Now, since $\cup\mathcal{S}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$, then there is a point $v \in \mathcal{X}$ such that $\{\mathcal{K}v\} = \mathcal{S}t$. Consequently, we have $\{t\} = \{\mathcal{J}t\} = \mathcal{S}t = \{\mathcal{K}v\}$. Again the use of (3.1) gives

$$\begin{aligned}\Phi(\delta(\mathcal{T}v, \mathcal{S}t)) \leq a(d(\mathcal{K}v, \mathcal{J}t))&\Phi(d(\mathcal{K}v, \mathcal{J}t))\\ &+ b(d(\mathcal{K}v, \mathcal{J}t))\left[\Phi(\delta(\mathcal{K}v, \mathcal{T}v)) + \Phi(\delta(\mathcal{J}t, \mathcal{S}t))\right]\\ &+ c(d(\mathcal{K}v, \mathcal{J}t))\min\left\{\Phi(D(\mathcal{K}v, \mathcal{S}t)), \Phi(D(\mathcal{J}t, \mathcal{T}v))\right\}.\end{aligned}$$

If we had $\mathcal{T}v \neq \{t\}$, then by properties (2.1) and (3.2) we would get

$$\begin{aligned}\Phi(\delta(\mathcal{T}v, t)) &\leq a(d(t, t))\Phi(d(t, t))\\ &\quad + b(d(t, t))\left[\Phi(\delta(t, \mathcal{T}v)) + \Phi(\delta(t, t))\right]\\ &\quad + c(d(t, t))\min\left\{\Phi(D(t, t)), \Phi(D(t, \mathcal{T}v))\right\}\\ &= b(0)\Phi(\delta(t, \mathcal{T}v)) < \Phi(\delta(t, \mathcal{T}v)).\end{aligned}$$

This is a contradiction, so we have $\mathcal{T}v = \{t\} = \{\mathcal{K}v\}$. Thus, $\{t\} = \{\mathcal{J}t\} = \mathcal{S}t = \{\mathcal{K}v\} = \mathcal{T}v$. Since $\mathcal{T}$ and $\mathcal{K}$ are weakly compatible, $\mathcal{T}v = \{\mathcal{K}v\}$ implies that $\mathcal{T}\mathcal{K}v = \mathcal{K}\mathcal{T}v$ and so $\mathcal{T}t = \{\mathcal{K}t\}$. We confirm that $\{t\} = \mathcal{T}t = \{\mathcal{K}t\}$. If not, then by (3.1) and conditions (2.1) and (3.2) it comes

$$\begin{aligned}\Phi(\delta(\mathcal{T}t, t)) = \Phi(\delta(\mathcal{T}t, \mathcal{S}t)) &\leq a(d(\mathcal{K}t, \mathcal{J}t))\Phi(d(\mathcal{K}t, \mathcal{J}t))\\ &\quad + b(d(\mathcal{K}t, \mathcal{J}t))\left[\Phi(\delta(\mathcal{K}t, \mathcal{T}t)) + \Phi(\delta(\mathcal{J}t, \mathcal{S}t))\right]\\ &\quad + c(d(\mathcal{K}t, \mathcal{J}t))\min\left\{\Phi(D(\mathcal{K}t, \mathcal{S}t)), \Phi(D(\mathcal{J}t, \mathcal{T}t))\right\}\\ &= a(d(\mathcal{T}t, t))\Phi(d(\mathcal{T}t, t)) + c(d(\mathcal{T}t, t))\Phi(D(\mathcal{T}t, t))\\ &\leq \left[a(d(\mathcal{T}t, t)) + c(d(\mathcal{T}t, t))\right]\Phi(\delta(\mathcal{T}t, t)) < \Phi(\delta(\mathcal{T}t, t)),\end{aligned}$$

which is a contradiction. Hence $\mathcal{T}t = \{\mathcal{K}t\} = \{t\}$. Consequently, $\{t\} = \{\mathcal{K}t\} = \{\mathcal{J}t\} = \mathcal{S}t = \mathcal{T}t$ and t is a common fixed point of $\mathcal{J}, \mathcal{K}, \mathcal{S}$ and $\mathcal{T}$. Similarly, one can obtain this conclusion by assuming $\mathcal{S}(\mathcal{X})$ is closed.

Finally, we prove that t is unique. Let t' be another common fixed point of maps $\mathcal{J}, \mathcal{K}, \mathcal{S}$ and $\mathcal{T}$ such that $t' \neq t$. Then, using inequality (3.1) and properties (2.1) and (3.2) we obtain

$$\begin{aligned}\Phi(\delta(t, t')) = \Phi(\delta(\mathcal{T}t, \mathcal{S}t')) &\leq a(d(\mathcal{K}t, \mathcal{J}t'))\Phi(d(\mathcal{K}t, \mathcal{J}t'))\\ &\quad + b(d(\mathcal{K}t, \mathcal{J}t'))\left[\Phi(\delta(\mathcal{K}t, \mathcal{T}t)) + \Phi(\delta(\mathcal{J}t', \mathcal{S}t'))\right]\end{aligned}$$

$$+ c(d(\mathcal{K}t, \mathcal{J}t')) \min\{\Phi(D(\mathcal{K}t, \mathcal{S}t')), \Phi(D(\mathcal{J}t', \mathcal{T}t))\}$$
$$= a(d(t,t'))\Phi(d(t,t')) + c(d(t,t'))\Phi(D(t,t'))$$
$$\leq [a(d(t,t')) + c(d(t,t'))]\,\Phi(\delta(t,t')) < \Phi(\delta(t,t')).$$

Therefore $t' = t$. Hence, t is the unique common fixed point of $\mathcal{J}, \mathcal{K}, \mathcal{S}$ and $\mathcal{T}$. □

If we put $\mathcal{S} = \mathcal{T}$ and $\mathcal{J} = \mathcal{K} = \mathcal{I}_{\mathcal{X}}$ (the identity map on $\mathcal{X}$) in Theorem 3.1 and we drop the closeness we get the next result.

Corollary 3.2. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{S}\colon \mathcal{X} \to B(\mathcal{X})$ be a set-valued map. Let Φ be as in Theorem 3.1. Assume that $\mathcal{S}$ satisfies the following inequality*

$$\Phi(\delta(\mathcal{S}x, \mathcal{S}y)) \leq a(d(x,y))\Phi(d(x,y))$$
$$+ b(d(x,y))\,[\Phi(\delta(x, \mathcal{S}x)) + \Phi(\delta(y, \mathcal{S}y))]$$
$$+ c(d(x,y)) \min\{\Phi(D(x, \mathcal{S}y)), \Phi(D(y, \mathcal{S}x))\}$$

for all $x, y \in \mathcal{X}$, where a, b and c are as in Theorem 3.1. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.

If we let in Theorem 3.1, $\mathcal{S} = \mathcal{T}$ and $\mathcal{J} = \mathcal{K}$, then we obtain the following result.

Corollary 3.3. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{S}\colon \mathcal{X} \to B(\mathcal{X})$, $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ be a set-valued map (resp. a single-valued map). Assume that $\mathcal{S}$ and $\mathcal{K}$ satisfy conditions*

(i) $\cup\mathcal{S}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$ *and*

(ii) *the inequality*

$$\Phi(\delta(\mathcal{S}x, \mathcal{S}y)) \leq a(d(\mathcal{K}x, \mathcal{K}y))\Phi(d(\mathcal{K}x, \mathcal{K}y))$$
$$+ b(d(\mathcal{K}x, \mathcal{K}y))\,[\Phi(\delta(\mathcal{K}x, \mathcal{S}x)) + \Phi(\delta(\mathcal{K}y, \mathcal{S}y))]$$
$$+ c(d(\mathcal{K}x, \mathcal{K}y)) \min\{\Phi(D(\mathcal{K}x, \mathcal{S}y)), \Phi(D(\mathcal{K}y, \mathcal{S}x))\}$$

holds for all $x, y \in \mathcal{X}$, where Φ, a, b and c are as in Theorem 3.1.

If maps $\mathcal{S}$ and $\mathcal{K}$ are weakly compatible and $\mathcal{S}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) is closed or $\mathcal{K}$ is surjective, then $\mathcal{S}$ and $\mathcal{K}$ possess a unique common fixed point in $\mathcal{X}$.

Now, if we put $\mathcal{J} = \mathcal{K} = \mathcal{I}_{\mathcal{X}}$, then we get the following corollary.

Corollary 3.4. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{S}, \mathcal{T} \colon \mathcal{X} \to B(\mathcal{X})$ be two set-valued maps such that*

$$\begin{aligned}\Phi(\delta(\mathcal{T}x, \mathcal{S}y)) \leq a(d(x,y))\Phi(d(x,y)) &\\ + b(d(x,y))\,[\Phi(\delta(x, \mathcal{T}x)) + \Phi(\delta(y, \mathcal{S}y))] &\\ + c(d(x,y))\min\{\Phi(D(x, \mathcal{S}y)), \Phi(D(y, \mathcal{T}x))\} &\end{aligned}$$

for all $x, y \in \mathcal{X}$, where Φ, a, b and c are as in Theorem 3.1. If $\mathcal{S}(\mathcal{X})$ or $\mathcal{T}(\mathcal{X})$ is closed, then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in $\mathcal{X}$.

Obviously, Theorem 3.1 is a generalization of the result of [2], since no continuity hypothesis is assumed here and the weak compatibility is among the least conditions for maps to have common fixed points.

Remark 3.5.

(1) From condition (3.3) it is easy to see that Theorem 3.1 remains valid if $\mathcal{J}$ or $\mathcal{K}$ is surjective in lieu of $\mathcal{S}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) or $\mathcal{T}(\mathcal{X})$ (resp. $\mathcal{J}(\mathcal{X})$) is closed.

(2) The same result remains valid if we replace inequality (3.1) by the following one

$$\begin{aligned}\Phi(\delta(\mathcal{T}x, \mathcal{S}y)) \leq a(d(\mathcal{K}x, \mathcal{J}y))\Phi(d(\mathcal{K}x, \mathcal{J}y)) &\\ + b(d(\mathcal{K}x, \mathcal{J}y))\,[\Phi(\delta(\mathcal{K}x, \mathcal{T}x)) + \Phi(\delta(\mathcal{J}y, \mathcal{S}y))] &\\ + c(d(\mathcal{K}x, \mathcal{J}y))\left[\frac{\Phi(D(\mathcal{K}x, \mathcal{S}y)) + \Phi(D(\mathcal{J}y, \mathcal{T}x))}{2}\right] &\end{aligned}$$

with Φ satisfying, in addition to the hypothesis of Theorem 3.1, the property $\Phi(2t) \leq 2\Phi(t),\ t \geq 0$.

For a set-valued map $\mathcal{S} \colon \mathcal{X} \to B(\mathcal{X})$ (resp. a self-map $\mathcal{J} \colon \mathcal{X} \to \mathcal{X}$), we denote $F_{\mathcal{S}} = \{x \in \mathcal{X} : \mathcal{S}(x) = \{x\}\}$ (resp. $F_{\mathcal{J}} = \{x \in \mathcal{X} : \mathcal{J}(x) = x\}$).

Theorem 3.6 (cf. [7, Theorem 3]). *Let $\mathcal{S}, \mathcal{T} \colon \mathcal{X} \to B(\mathcal{X})$ be set-valued maps and $\mathcal{J}, \mathcal{K} \colon \mathcal{X} \to \mathcal{X}$ be self-maps on the metric space $\mathcal{X}$. If inequality* (3.1) *holds for all $x, y \in \mathcal{X}$ with Φ, a, b, c satisfying* (2.1) *and* (3.2)*, then*

$$(F_{\mathcal{K}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{S}} = (F_{\mathcal{K}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{T}}.$$

Proof. Let $u \in (F_{\mathcal{K}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{S}}$. If we had $u \notin F_{\mathcal{T}}$, then by estimation (3.1) and properties (2.1) and (3.2) we would get

$$\begin{aligned}\Phi(\delta(\mathcal{T}u, u)) = \Phi(\delta(\mathcal{T}u, \mathcal{S}u)) &\le a(d(\mathcal{K}u, \mathcal{J}u))\Phi(d(\mathcal{K}u, \mathcal{J}u)) \\ &\quad + b(d(\mathcal{K}u, \mathcal{J}u))\left[\Phi(\delta(\mathcal{K}u, \mathcal{T}u)) + \Phi(\delta(\mathcal{J}u, \mathcal{S}u))\right] \\ &\quad + c(d(\mathcal{K}u, \mathcal{J}u))\min\{\Phi(D(\mathcal{K}u, \mathcal{S}u)), \Phi(D(\mathcal{J}u, \mathcal{T}u))\} \\ &= b(0)\Phi(\delta(u, \mathcal{T}u)) < \Phi(\delta(u, \mathcal{T}u)).\end{aligned}$$

This contradiction implies that $\mathcal{T}u = \{u\}$. Thus,

$$(F_{\mathcal{K}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{S}} \subset (F_{\mathcal{K}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{T}}.$$

Similarly, we can prove that

$$(F_{\mathcal{K}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{T}} \subset (F_{\mathcal{K}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{S}}. \qquad \square$$

Theorems 3.1 and 3.6 imply the following one.

Theorem 3.7. *Let $(\mathcal{X}, d)$ be a complete metric space. Let $\mathcal{J}, \mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ be two self-maps and $\mathcal{S}_i\colon \mathcal{X} \to B(\mathcal{X})$, $i \in \mathbb{N}^*$, be set-valued maps such that*

(i) $\cup\mathcal{S}_i(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$ *and* $\cup\mathcal{S}_{i+1}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$,

(ii) *either $\mathcal{S}_i(\mathcal{X})$(resp. $\mathcal{J}(\mathcal{X})$) or $\mathcal{S}_{i+1}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) is closed,*

(iii) *the pairs $\{\mathcal{S}_i, \mathcal{K}\}$ and $\{\mathcal{S}_{i+1}, \mathcal{J}\}$ are weakly compatible.*

Let Φ be an increasing and continuous function of $[0, \infty)$ into itself satisfying (2.1) *and the inequality*

$$\begin{aligned}\Phi(\delta(\mathcal{S}_i x, \mathcal{S}_{i+1} y)) &\le a(d(\mathcal{K}x, \mathcal{J}y))\Phi(d(\mathcal{K}x, \mathcal{J}y)) \\ &\quad + b(d(\mathcal{K}x, \mathcal{J}y))\left[\Phi(\delta(\mathcal{K}x, \mathcal{S}_i x)) + \Phi(\delta(\mathcal{J}y, \mathcal{S}_{i+1} y))\right] \\ &\quad + c(d(\mathcal{K}x, \mathcal{J}y))\min\{\Phi(D(\mathcal{K}x, \mathcal{S}_{i+1} y)), \Phi(D(\mathcal{J}y, \mathcal{S}_i x))\}\end{aligned}$$

holds for all $x, y \in \mathcal{X}$, $i \in \mathbb{N}^*$, where $a, b, c\colon [0, \infty) \to [0, 1)$ are continuous increasing functions satisfying (3.2). *Then $\mathcal{J}, \mathcal{K}$ and $\{\mathcal{S}_i\}_{i \in \mathbb{N}^*}$ have a unique common fixed point in $\mathcal{X}$.*

Remark 3.8. Theorem 3.7 remains valid if $\mathcal{J}$ or $\mathcal{K}$ is surjective in lieu of the condition (ii).

Now, we establish a fixed point theorem under a strict contractive condition in a metric space. Our version requires neither continuity nor compactness but only minimal conditions and a concept of maps called D-maps.

Definition 3.9 ([1]). Maps $\mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ are said to be D-maps if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{K}x_n = t$ and $\lim_{n\to\infty} \mathcal{T}x_n = \{t\}$ for some $t \in \mathcal{X}$.

Example 3.10.

(1) Consider $\mathcal{X} = [0, \infty)$ with the usual metric and define $\mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}x = [0, x] \quad \text{and} \quad \mathcal{K}x = 2x, \ \forall x \in \mathcal{X}.$$

Let $x_n = \frac{1}{3n}$ for all $n \in \mathbb{N}^*$. Clearly, $\lim_{n\to\infty} \mathcal{T}x_n = \{0\}$ and $\lim_{n\to\infty} \mathcal{K}x_n = 0$. That is, $\mathcal{T}$ and $\mathcal{K}$ are D-maps.

(2) Consider $\mathcal{X} = [0, \infty)$ with the usual metric and define $\mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}x = [0, x] \quad \text{and} \quad \mathcal{K}x = 3x + 2, \ \forall x \in \mathcal{X}.$$

Suppose there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{K}x_n = t$ and $\lim_{n\to\infty} y_n = t$ for some $t \in [0, \infty)$, with $y_n \in \mathcal{T}x_n = [0, x_n]$. Then $\lim_{n\to\infty} x_n = \frac{t-2}{3}$ and $0 \le t \le \frac{t-2}{3}$, which is impossible. Thus $\mathcal{T}$ and $\mathcal{K}$ are not D-maps.

Theorem 3.11. *Let $\mathcal{J}, \mathcal{K}$ be single-valued maps from a metric space $(\mathcal{X}, d)$ into itself and $\mathcal{S}, \mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ be two set-valued maps with $\cup\mathcal{T}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$ and $\cup\mathcal{S}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$. Let Φ be an increasing and continuous function of $[0, \infty)$ into itself satisfying (2.1). Suppose that inequality (3.1) holds for all $x, y \in \mathcal{X}$, where functions $a, b, c\colon [0, \infty) \to [0, 1)$ are only continuous and satisfy (3.2). If either*

(3') *$\mathcal{T}, \mathcal{K}$ are weakly compatible D-maps; $\mathcal{S}, \mathcal{J}$ are weakly compatible and $\mathcal{T}(\mathcal{X})$ (resp. $\mathcal{J}(\mathcal{X})$) is closed or*

(3”) *$\mathcal{S}, \mathcal{J}$ are weakly compatible D-maps; $\mathcal{T}, \mathcal{K}$ are weakly compatible and $\mathcal{S}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) is closed,*

then there is a unique common fixed point $t \in \mathcal{X}$, i.e.

$$\mathcal{S}t = \mathcal{T}t = \{t\} = \{\mathcal{J}t\} = \{\mathcal{K}t\}.$$

Proof. Suppose that $\mathcal{T}$ and $\mathcal{K}$ are D-maps, then there is a sequence $\{x_n\}$ in $\mathcal{X}$ such that, $\lim_{n\to\infty} \mathcal{K}x_n = t$ and $\lim_{n\to\infty} \mathcal{T}x_n = \{t\}$ for some $t \in \mathcal{X}$. Since $\mathcal{T}(\mathcal{X})$ is closed we have $\{t\} \in \mathcal{T}(\mathcal{X})$. Consequently, $t \in \cup\mathcal{T}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$. It then follows that there exists a point u in $\mathcal{X}$ such that $\mathcal{J}u = t$. By condition (3.1) we have

$$\begin{aligned}\Phi(\delta(\mathcal{T}x_n, \mathcal{S}u)) \leq\ & a(d(\mathcal{K}x_n, \mathcal{J}u))\Phi(d(\mathcal{K}x_n, \mathcal{J}u)) \\ & + b(d(\mathcal{K}x_n, \mathcal{J}u))\left[\Phi(\delta(\mathcal{K}x_n, \mathcal{T}x_n)) + \Phi(\delta(\mathcal{J}u, \mathcal{S}u))\right] \\ & + c(d(\mathcal{K}x_n, \mathcal{J}u)) \min\left\{\Phi(D(\mathcal{K}x_n, \mathcal{S}u)), \Phi(D(\mathcal{J}u, \mathcal{T}x_n))\right\}.\end{aligned}$$

If we had $\mathcal{S}u \neq \{\mathcal{J}u\}$, then letting $n \to \infty$, by the continuity of the functions Φ, a, b and c, using Lemma 2.1 and properties (2.1) and (3.2), we would obtain

$$\begin{aligned}\Phi(\delta(\mathcal{J}u, \mathcal{S}u)) \leq\ & a(d(\mathcal{J}u, \mathcal{J}u))\Phi(d(\mathcal{J}u, \mathcal{J}u)) \\ & + b(d(\mathcal{J}u, \mathcal{J}u))\left[\Phi(\delta(\mathcal{J}u, \mathcal{J}u)) + \Phi(\delta(\mathcal{J}u, \mathcal{S}u))\right] \\ & + c(d(\mathcal{J}u, \mathcal{J}u)) \min\left\{\Phi(D(\mathcal{J}u, \mathcal{S}u)), \Phi(D(\mathcal{J}u, \mathcal{J}u))\right\} \\ =\ & b(0)\Phi(\delta(\mathcal{J}u, \mathcal{S}u)) < \Phi(\delta(\mathcal{J}u, \mathcal{S}u)),\end{aligned}$$

which is a contradiction. Thus, $\mathcal{S}u = \{\mathcal{J}u\}$. Hence, by the weak compatibility we get, $\mathcal{S}\mathcal{S}u = \mathcal{S}\mathcal{J}u = \mathcal{J}\mathcal{S}u = \{\mathcal{J}\mathcal{J}u\}$. Again, by (3.1), we have

$$\begin{aligned}\Phi(\delta(\mathcal{T}x_n, \mathcal{S}\mathcal{S}u)) \leq\ & a(d(\mathcal{K}x_n, \mathcal{J}\mathcal{S}u))\Phi(d(\mathcal{K}x_n, \mathcal{J}\mathcal{S}u)) \\ & + b(d(\mathcal{K}x_n, \mathcal{J}\mathcal{S}u))\left[\Phi(\delta(\mathcal{K}x_n, \mathcal{T}x_n)) + \Phi(\delta(\mathcal{J}\mathcal{S}u, \mathcal{S}\mathcal{S}u))\right] \\ & + c(d(\mathcal{K}x_n, \mathcal{J}\mathcal{S}u)) \min\left\{\Phi(D(\mathcal{K}x_n, \mathcal{S}\mathcal{S}u)),\ \Phi(D(\mathcal{J}\mathcal{S}u, \mathcal{T}x_n))\right\}.\end{aligned}$$

If we had $\mathcal{S}\mathcal{S}u \neq \{\mathcal{J}u\}$, then letting $n \to \infty$, since Φ is increasing, by the continuity of Φ, a, b and c, the use of Lemma 2.1 and conditions (2.1) and (3.2), we would obtain

$$\begin{aligned}\Phi(\delta(\mathcal{J}u, \mathcal{S}\mathcal{S}u)) \leq\ & a(d(\mathcal{J}u, \mathcal{J}\mathcal{S}u))\Phi(d(\mathcal{J}u, \mathcal{J}\mathcal{S}u)) \\ & + b(d(\mathcal{J}u, \mathcal{J}\mathcal{S}u))\left[\Phi(\delta(\mathcal{J}u, \mathcal{J}u)) + \Phi(\delta(\mathcal{J}\mathcal{S}u, \mathcal{S}\mathcal{S}u))\right] \\ & + c(d(\mathcal{J}u, \mathcal{J}\mathcal{S}u)) \min\left\{\Phi(D(\mathcal{J}u, \mathcal{S}\mathcal{S}u)), \Phi(D(\mathcal{J}\mathcal{S}u, \mathcal{J}u))\right\} \\ =\ & a(d(\mathcal{J}u, \mathcal{S}\mathcal{S}u))\Phi(d(\mathcal{J}u, \mathcal{S}\mathcal{S}u)) \\ & + c(d(\mathcal{J}u, \mathcal{S}\mathcal{S}u))\Phi(D(\mathcal{J}u, \mathcal{S}\mathcal{S}u)) \\ \leq\ & \left[a(d(\mathcal{J}u, \mathcal{S}\mathcal{S}u)) + c(d(\mathcal{J}u, \mathcal{S}\mathcal{S}u))\right]\Phi(\delta(\mathcal{J}u, \mathcal{S}\mathcal{S}u)) \\ <\ & \Phi(\delta(\mathcal{J}u, \mathcal{S}\mathcal{S}u)).\end{aligned}$$

This is a contradiction, so we have $\mathcal{S}\mathcal{S}u = \mathcal{J}\mathcal{S}u = \{\mathcal{J}u\}$, i.e. $\mathcal{S}\mathcal{S}u = \mathcal{J}\mathcal{S}u = \mathcal{S}u$ and $\mathcal{S}u$ is a common fixed point of $\mathcal{S}$ and $\mathcal{J}$. Since $\cup\mathcal{S}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$, then, there is a point $v \in \mathcal{X}$ such that $\{\mathcal{K}v\} = \mathcal{S}u$. If we had $\mathcal{T}v \neq \{\mathcal{K}v\}$, then by condition (3.1) and properties (2.1) and (3.2) we would have

$$\begin{aligned}\Phi(\delta(\mathcal{T}v, \mathcal{K}v)) = \Phi(\delta(\mathcal{T}v, \mathcal{S}u))&\\ &\leq a(d(\mathcal{K}v, \mathcal{J}u))\Phi(d(\mathcal{K}v, \mathcal{J}u))\\ &\quad + b(d(\mathcal{K}v, \mathcal{J}u))\left[\Phi(\delta(\mathcal{K}v, \mathcal{T}v)) + \Phi(\delta(\mathcal{J}u, \mathcal{S}u))\right]\\ &\quad + c(d(\mathcal{K}v, \mathcal{J}u))\min\left\{\Phi(D(\mathcal{K}v, \mathcal{S}u)), \Phi(D(\mathcal{J}u, \mathcal{T}v))\right\}\\ &= b(0)\Phi(\delta(\mathcal{K}v, \mathcal{T}v)) < \Phi(\delta(\mathcal{K}v, \mathcal{T}v)).\end{aligned}$$

This is a contradiction, thus $\mathcal{T}v = \{\mathcal{K}v\} = \mathcal{S}u$. By the weak compatibility of $\mathcal{T}$ and $\mathcal{K}$ we have $\mathcal{T}\mathcal{T}v = \mathcal{T}\mathcal{K}v = \mathcal{K}\mathcal{T}v = \{\mathcal{K}\mathcal{K}v\}$. Again, if we had $\mathcal{T}\mathcal{T}v \neq \mathcal{S}u$, then, since Φ is increasing, by conditions (3.1), (2.1) and (3.2), we would have

$$\begin{aligned}\Phi(\delta(\mathcal{T}\mathcal{T}v, \mathcal{S}u)) &\leq a(d(\mathcal{K}\mathcal{T}v, \mathcal{J}u))\Phi(d(\mathcal{K}\mathcal{T}v, \mathcal{J}u))\\ &\quad + b(d(\mathcal{K}\mathcal{T}v, \mathcal{J}u))\left[\Phi(\delta(\mathcal{K}\mathcal{T}v, \mathcal{T}\mathcal{T}v)) + \Phi(\delta(\mathcal{J}u, \mathcal{S}u))\right]\\ &\quad + c(d(\mathcal{K}\mathcal{T}v, \mathcal{J}u))\min\left\{\Phi(D(\mathcal{K}\mathcal{T}v, \mathcal{S}u)), \Phi(D(\mathcal{J}u, \mathcal{T}\mathcal{T}v))\right\}\\ &= a(d(\mathcal{T}\mathcal{T}v, \mathcal{S}u))\Phi(d(\mathcal{T}\mathcal{T}v, \mathcal{S}u))\\ &\quad + c(d(\mathcal{T}\mathcal{T}v, \mathcal{S}u))\Phi(D(\mathcal{T}\mathcal{T}v, \mathcal{S}u))\\ &\leq \left[a(d(\mathcal{T}\mathcal{T}v, \mathcal{S}u)) + c(d(\mathcal{T}\mathcal{T}v, \mathcal{S}u))\right]\Phi(\delta(\mathcal{T}\mathcal{T}v, \mathcal{S}u))\\ &< \Phi(\delta(\mathcal{T}\mathcal{T}v, \mathcal{S}u)).\end{aligned}$$

This contradiction shows that $\mathcal{T}\mathcal{T}v = \mathcal{S}u$, i.e., $\mathcal{T}\mathcal{S}u = \mathcal{S}u = \mathcal{K}\mathcal{S}u$ and $\mathcal{S}u$ is also a common fixed point of $\mathcal{T}$ and $\mathcal{K}$. Since $\mathcal{S}u = \{t\}$, then

$$\mathcal{S}t = \mathcal{T}t = \{t\} = \{\mathcal{K}t\} = \{\mathcal{J}t\}.$$

Finally, we prove that t is unique. Indeed, let $t' \neq t$ be another common fixed point of the maps $\mathcal{J}, \mathcal{K}, \mathcal{S}$ and $\mathcal{T}$. Since Φ is increasing, by estimation (3.1) and conditions (2.1) and (3.2), one may get

$$\begin{aligned}\Phi(d(t, t')) = \Phi(\delta(\mathcal{T}t, \mathcal{S}t')) &\leq a(d(\mathcal{K}t, \mathcal{J}t'))\Phi(d(\mathcal{K}t, \mathcal{J}t'))\\ &\quad + b(d(\mathcal{K}t, \mathcal{J}t'))\left[\Phi(\delta(\mathcal{K}t, \mathcal{T}t)) + \Phi(\delta(\mathcal{J}t', \mathcal{S}t'))\right]\end{aligned}$$

$$+ c(d(\mathcal{K}t, \mathcal{J}t')) \min \{\Phi(D(\mathcal{K}t, \mathcal{S}t')), \Phi(D(\mathcal{J}t', \mathcal{T}t))\}$$

$$= a(d(t,t'))\Phi(d(t,t')) + c(d(t,t'))\Phi(D(t,t'))$$

$$\leq [a(d(t,t')) + c(d(t,t'))]\,\Phi(d(t,t')) < \Phi(d(t,t')).$$

This contradiction implies that $t' = t$.

Similarly, one can obtain this conclusion by using (3”) in lieu of (3’). □

REMARK 3.12. Theorem 3.11 remains valid if we replace inequality (3.1) by

$$\Phi(\delta(\mathcal{T}x, \mathcal{S}y)) \leq a(d(\mathcal{K}x, \mathcal{J}y))\Phi(d(\mathcal{K}x, \mathcal{J}y))$$
$$+ b(d(\mathcal{K}x, \mathcal{J}y))\,[\Phi(\delta(\mathcal{K}x, \mathcal{T}x)) + \Phi(\delta(\mathcal{J}y, \mathcal{S}y))]$$
$$+ c(d(\mathcal{K}x, \mathcal{J}y)) \left[\frac{\Phi(D(\mathcal{K}x, \mathcal{S}y)) + \Phi(D(\mathcal{J}y, \mathcal{T}x))}{2}\right],$$

where – in addition to the hypothesis of Theorem 3.1 – Φ satisfies also the condition $\Phi(2t) \leq 2\Phi(t),\ t \geq 0$, or

$$\delta(\mathcal{T}x, \mathcal{S}y) \leq \alpha \max\{d(\mathcal{K}x, \mathcal{J}y), \delta(\mathcal{K}x, \mathcal{T}x), \delta(\mathcal{J}y, \mathcal{S}y)\}$$
$$+ (1-\alpha)\,[aD(\mathcal{K}x, \mathcal{S}y) + bD(\mathcal{J}y, \mathcal{T}x)],$$

where $0 \leq \alpha < 1, 0 \leq a \leq \frac{1}{2}$ and $0 \leq b < \frac{1}{2}$.

In Theorem 3.11, if we set $\mathcal{S} = \mathcal{T}$ and $\mathcal{K} = \mathcal{J}$, then we will get the following result.

COROLLARY 3.13. *Let $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ be a single-valued map of a metric space $(\mathcal{X}, d)$ and $\mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ be a set-valued map. Assume that $\mathcal{T}$ and $\mathcal{K}$ satisfy the conditions*

(i) $\cup\mathcal{T}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$,

(ii) *the inequality*

$$\Phi(\delta(\mathcal{T}x, \mathcal{T}y)) \leq a(d(\mathcal{K}x, \mathcal{K}y))\Phi(d(\mathcal{K}x, \mathcal{K}y))$$
$$+ b(d(\mathcal{K}x, \mathcal{K}y))\,[\Phi(\delta(\mathcal{K}x, \mathcal{T}x)) + \Phi(\delta(\mathcal{K}y, \mathcal{T}y))]$$
$$+ c(d(\mathcal{K}x, \mathcal{K}y)) \min\{\Phi(D(\mathcal{K}x, \mathcal{T}y)), \Phi(D(\mathcal{K}y, \mathcal{T}x))\}$$

holds for all $x, y \in \mathcal{X}$, where Φ is as in Theorem 3.1 and functions a, b and c are as in Theorem 3.11.

If $\mathcal{T}$ and $\mathcal{K}$ are weakly compatible D-maps and $\mathcal{T}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) is closed, then there is a unique common fixed point t in $\mathcal{X}$, i.e.

$$\mathcal{T}t = \{t\} = \{\mathcal{K}t\}.$$

For three maps we have the following result.

COROLLARY 3.14. *Let $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}$ be a single-valued map of a metric space $(\mathcal{X}, d)$ and $\mathcal{S}, \mathcal{T}\colon \mathcal{X} \to B(\mathcal{X})$ be two set-valued maps such that*

(i) $\cup\mathcal{T}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$ *and* $\cup\mathcal{S}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$,

(ii) *the inequality*

$$\begin{aligned}\Phi(\delta(\mathcal{T}x, \mathcal{S}y)) \leq a(d(\mathcal{K}x, \mathcal{K}y))\Phi(d(\mathcal{K}x, \mathcal{K}y)) \\ + b(d(\mathcal{K}x, \mathcal{K}y))\,[\Phi(\delta(\mathcal{K}x, \mathcal{T}x)) + \Phi(\delta(\mathcal{K}y, \mathcal{S}y))] \\ + c(d(\mathcal{K}x, \mathcal{K}y))\min\{\Phi(D(\mathcal{K}x, \mathcal{S}y)), \Phi(D(\mathcal{K}y, \mathcal{T}x))\}\end{aligned}$$

holds for all $x, y \in \mathcal{X}$, where Φ is as in Theorem 3.1 and functions a, b and c are as in Theorem 3.11. If either

(iii) *$\mathcal{T}, \mathcal{K}$ are weakly compatible D-maps; $\mathcal{S}, \mathcal{K}$ are weakly compatible and $\mathcal{T}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) is closed or*

(iv) *$\mathcal{S}, \mathcal{K}$ are weakly compatible D-maps; $\mathcal{T}, \mathcal{K}$ are weakly compatible and $\mathcal{S}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) is closed,*

then there is a unique common fixed point t in $\mathcal{X}$, i.e.

$$\mathcal{S}t = \mathcal{T}t = \{\mathcal{K}t\} = \{t\}.$$

Now, we give a generalization of Theorem 3.11.

THEOREM 3.15. *Let $\mathcal{J}, \mathcal{K}$ be single-valued maps of a metric space $(\mathcal{X}, d)$ and $\mathcal{S}_n\colon \mathcal{X} \to B(\mathcal{X}), n \in \mathbb{N}^*$ be set-valued maps such that*

(i) $\cup\mathcal{S}_n(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$ *and* $\cup\mathcal{S}_{n+1}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$,

(ii) *the inequality*

$$\begin{aligned}\Phi(\delta(\mathcal{S}_n x, \mathcal{S}_{n+1} y)) \leq a(d(\mathcal{K}x, \mathcal{J}y))\Phi(d(\mathcal{K}x, \mathcal{J}y)) \\ + b(d(\mathcal{K}x, \mathcal{J}y))\,[\Phi(\delta(\mathcal{K}x, \mathcal{S}_n x)) + \Phi(\delta(\mathcal{J}y, \mathcal{S}_{n+1} y))] \\ + c(d(\mathcal{K}x, \mathcal{J}y))\min\{\Phi(D(\mathcal{K}x, \mathcal{S}_{n+1} y)), \Phi(D(\mathcal{J}y, \mathcal{S}_n x))\}\end{aligned}$$

holds for all $x, y \in \mathcal{X}, n \in \mathbb{N}^*$, where Φ is as in Theorem 3.1 and functions a, b and c are as in Theorem 3.11. If either

(iii) $\mathcal{K}$ *and* $\{\mathcal{S}_n\}_{n\in\mathbb{N}^*}$ *are weakly compatible D-maps;* $\mathcal{J}$ *and* $\{\mathcal{S}_{n+1}\}_{n\in\mathbb{N}^*}$ *are weakly compatible and* $\mathcal{S}_n(\mathcal{X})$ *(resp.* $\mathcal{J}(\mathcal{X})$*) is closed or*

(iv) $\mathcal{J}$ *and* $\{\mathcal{S}_{n+1}\}_{n\in\mathbb{N}^*}$ *are weakly compatible D-maps;* $\mathcal{K}$ *and* $\{\mathcal{S}_n\}_{n\in\mathbb{N}^*}$ *are weakly compatible and* $\mathcal{S}_{n+1}(\mathcal{X})$ *(resp.* $\mathcal{K}(\mathcal{X})$*) is closed,*

then there is a unique common fixed point $t \in \mathcal{X}$*, i.e.*

$$\mathcal{S}_n t = \{t\} = \{\mathcal{J}t\} = \{\mathcal{K}t\}, \quad n \in \mathbb{N}^*.$$

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