DE GRUYTER
OPEN
Annales Mathematicae Silesianae **32** (2018), 275–284
DOI: 10.1515/amsil-2017-0011

SOME NEW OSTROWSKI'S INEQUALITIES FOR FUNCTIONS WHOSE n^{th}DERIVATIVES ARE LOGARITHMICALLY CONVEX

Badreddine Meftah

Abstract. Some new Ostrowski's inequalities for functions whose n^{th}derivative are logarithmically convex are established.

1. Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1 ([2]). *Let $f\colon I\to\mathbb{R}$, where $I\subseteq\mathbb{R}$ is an interval, be a differentiable mapping in the interior I° of I, and $a,b\in I^{\circ}$, with $a<b$. If $|f'|\le M$ for all $x\in[a,b]$, then*

$$(1.1)\quad \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le M(b-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right],\quad \forall x\in[a,b].$$

This is well-known as Ostrowski inequality. In recent years, a number of authors have written about generalizations, extensions and variants of inequality (1.1).

Received: 29.01.2017. Accepted: 20.07.2017. Published online: 28.11.2017.
(2010) Mathematics Subject Classification: 26D10, 26D15, 26A51.
Key words and phrases: Ostrowski inequality, Hölder inequality, power mean inequality, logarithmically convex functions.

In [1], Cerone et al. proved the following identity

Lemma 1 ([1, Lemma 2.1]). *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$ we have the identity*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) + (-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt,$$

where the kernel $K_n\colon [a,b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a,x], \\ \frac{(t-b)^n}{n!} & \text{if } t \in (x,b], \end{cases} \quad x \in [a,b],$$

and n is natural number, $n \geq 1$.

Wang et al. [4], proved the following identities

Lemma 2 ([4, Lemma 2.2]). *For $\alpha > 0$ and $k > 0$, $z > 0$:*

$$J(\alpha,k) := \int_0^1 (1-t)^{\alpha-1} k^t\,dt = \sum_{i=1}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha)_i} < \infty, \tag{1.2}$$

$$H(\alpha,k,z) := \int_0^1 t^{\alpha-1} k^t\,dt = z^\alpha k^z \sum_{i=1}^{\infty} \frac{(-z\ln k)^{i-1}}{(\alpha)_i} < \infty, \tag{1.3}$$

where $(\alpha)_i = \prod_{j=0}^{i-1} (\alpha+j)$.

We also recall that a positive function $f\colon I \to \mathbb{R}$ is said to be logarithmically convex, if

$$f(tx+(1-t)y) \leq [f(x)]^t [f(y)]^{1-t}$$

holds for all $x, y \in I$ and $t \in [0,1]$ (see [3]).

In this paper, by using the identity given in Lemma 1, we establish some new Ostrowski's inequalities for functions whose n^{th}derivatives are logarithmically convex.

2. Main results

In what follows, we assume that $n \in \mathbb{N}$, and $I \subset \mathbb{R}$ is an interval where $[a, b] \subset I$.

THEOREM 2. *Let $f\colon I \to \mathbb{R}$ be n times differentiable mapping on $[a, b]$ such that $f^{(n)} \in L([a, b])$ and $f^{(n)}(x) \neq 0$ for all $x \in [a, b]$. If $\left|f^{(n)}\right|$ is logarithmically convex, then the following inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| & \text{if } \lambda = \tau = 1 \\ \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i} & \text{if } \lambda = 1 \neq \tau \\ \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| + \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} & \text{if } \lambda \neq 1 = \tau \\ \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} & \\ \qquad + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i} & \text{if } \lambda \neq 1 \text{ and } \tau \neq 1 \end{cases}$$

holds for all $x \in [a, b]$, where

$$\lambda = \frac{\left|f^{(n)}(x)\right|}{\left|f^{(n)}(a)\right|}, \tag{2.1}$$

$$\tau = \frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(x)\right|}, \tag{2.2}$$

and $(n+1)_i = \prod_{j=0}^{i-1}(n+1+j)$.

PROOF. From Lemma 1 and the properties of modulus, we have

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right| du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right| du$$

$$(2.3) \qquad = \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n \left|f^{(n)}((1-t)\,a+tx)\right| dt + \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)\,x+tb)\right| dt.$$

Since $\left|f^{(n)}\right|$ is logarithmically convex, (2.3) becomes

$$(2.4) \qquad \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n \left|f^{(n)}(a)\right|^{1-t}\left|f^{(n)}(x)\right|^t dt + \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}(x)\right|^{1-t}\left|f^{(n)}(b)\right|^t dt$$

$$= \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right|\int_0^1 t^n\lambda^t\,dt + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right|\int_0^1 (1-t)^n\tau^t\,dt,$$

where λ and τ are defined as in (2.1) and (2.2) respectively.

Now, we proceed to the discussion of possible cases.

If $\lambda=\tau=1$, then (2.4) gives

$$(2.5) \qquad \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right|\int_0^1 t^n dt + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right|\int_0^1 (1-t)^n\,dt$$

$$= \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|.$$

If $\lambda=1$ and $\tau\neq1$, using (1.2), from (2.4) we obtain

$$(2.6) \qquad \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right|\sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i}.$$

If $\lambda \neq 1$ and $\tau = 1$, we can use (1.3) with $z = 1$. Then, from (2.4) we obtain

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \tag{2.7}$$
$$\leq \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| + \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda \sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i}.$$

In the case $\lambda \neq 1$ and $\tau \neq 1$, using Lemma 2 for (2.4), we get

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \tag{2.8}$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda \sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i}.$$

The desired result follows from (2.5)–(2.8). □

THEOREM 3. *Let $f\colon I \to \mathbb{R}$ be n times differentiable mapping on $[a,b]$ such that $f^{(n)} \in L([a,b])$ and $f^{(n)}(x) \neq 0$ for all $x \in [a,b]$, and let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f^{(n)}|^q$ is logarithmically convex, then the following inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \begin{cases} \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(x)\right| & \text{if } \lambda = \tau = 1 \\ \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\tau^q-1}{q\ln\tau}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \text{if } \lambda = 1 \neq \tau \\ \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(x)\right| & \text{if } \lambda \neq 1 = \tau \\ \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| & \\ \quad + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\tau^q-1}{q\ln\tau}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \text{if } \lambda \neq 1 \text{ and } \tau \neq 1 \end{cases}$$

holds for all $x \in [a,b]$, where λ and τ are defined by (2.1) and (2.2) respectively.

Proof. From Lemma 1, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\
&\le \int_a^x\frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right|du+\int_x^b\frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right|du\\
&=\frac{(x-a)^{n+1}}{n!}\int_0^1 t^n\left|f^{(n)}((1-t)\,a+tx)\right|dt\\
&\quad+\frac{(b-x)^{n+1}}{n!}\int_0^1(1-t)^n\left|f^{(n)}((1-t)\,x+tb)\right|dt\\
&\le\frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1(1-t)^{np}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}\\
&=\frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\int_0^1\left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\int_0^1\left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is logarithmically convex function, we deduce

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\
&\le\frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(a)\right|\left(\int_0^1\lambda^{qt}\,dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(x)\right|\left(\int_0^1\tau^{qt}\,dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.9}$$

Similarly to the proof of Theorem 2, if $\lambda=\tau=1$, then (2.9) gives

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \le \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(x)\right|. \tag{2.10}$$

If $\lambda=1$ and $\tau\neq 1$, then (2.9) becomes

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.11}$$
$$\le \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\int_0^1\tau^{qt}dt\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right|$$
$$= \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\frac{\tau^q-1}{q\ln\tau}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right|.$$

If $\lambda\neq 1$ and $\tau=1$, then (2.9) becomes

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.12}$$
$$\le \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(x)\right|+\frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right|$$
$$= \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(x)\right|+\frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right|.$$

In the case $\lambda\neq 1$ and $\tau\neq 1$, (2.9) gives

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.13}$$
$$\le \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\frac{\tau^q-1}{q\ln\tau}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right|.$$

The desired result follows from (2.10)–(2.13). □

Theorem 4. *Let $f : I \to \mathbb{R}$ be n times differentiable mapping on $[a,b]$ such that $f^{(n)} \in L([a,b])$ and $f^{(n)}(x) \neq 0$ for all $x \in [a,b]$, and let $q > 1$. If $\left|f^{(n)}\right|^q$ is logarithmically convex, then the following inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| & \text{if } \lambda = \tau = 1 \\ \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\sum_{i=1}^{\infty}\frac{(\ln \tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \text{if } \lambda = 1 \neq \tau \\ \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln \lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| & \text{if } \lambda \neq 1 = \tau \\ \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln \lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| \\ \quad + \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\sum_{i=1}^{\infty}\frac{(\ln \tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \text{if } \lambda \neq 1 \text{ and } \tau \neq 1 \end{cases}$$

holds for all $x \in [a,b]$, where λ and τ are defined by (2.1) and (2.2) respectively.

Proof. From Lemma 1, properties of modulus, and Hölder's inequality, we have

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right| du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right| du$$

$$= \frac{(x-a)^{n+1}}{n!}\int_0^1 (t^n)^{1-\frac{1}{q}} \cdot (t^n)^{\frac{1}{q}}\left|f^{(n)}((1-t)a + tx)\right| dt$$

$$+ \frac{(b-x)^{n+1}}{n!}\int_0^1 ((1-t)^n)^{1-\frac{1}{q}} \cdot ((1-t)^n)^{\frac{1}{q}}\left|f^{(n)}((1-t)x + tb)\right| dt$$

$$\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^n\left|f^{(n)}((1-t)a + tx)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^n\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}$$

$$=\frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left(\int_0^1 t^n \left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\ \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is logarithmically convex, we deduce that

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.14}$$

$$\leq \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left(\left|f^{(n)}(a)\right|^q\int_0^1 t^n\lambda^{qt}dt\right)^{\frac{1}{q}}$$

$$+\ \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\left|f^{(n)}(x)\right|^q\int_0^1 (1-t)^n\,\tau^{qt}dt\right)^{\frac{1}{q}}.$$

In the case $\lambda=\tau=1$, (2.14) gives

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.15}$$

$$\leq \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|.$$

If $\lambda=1$ and $\tau\neq 1$ then, applying (1.2) to (2.14), we get

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.16}$$

$$\leq \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\sum_{i=1}^{\infty}\frac{(\ln\tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right|.$$

If $\lambda\neq 1$ and $\tau=1$ then, applying equality (1.3) with $z=1$ to the integral $\int_0^1 t^n\left(\lambda^q\right)^t dt$, we get

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.17}$$
$$\leq \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln\lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}} + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|.$$

For $\lambda \neq 1$ and $\tau \neq 1$, using Lemma 2 for (2.14), we obtain

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.18}$$
$$\leq \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln\lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}$$
$$+ \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|\left(\sum_{i=1}^{\infty}\frac{(\ln\tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}.$$

The desired result follows from (2.15)–(2.18). The proof is thus completed. □

References

[1] Cerone P., Dragomir S.S., Roumeliotis J., *Some Ostrowski type inequalities for n-time differentiable mappings and applications*, Demonstratio Math. **32** (1999), no. 4, 697–712.

[2] Mitrinović D.S., Pečarić J.E., Fink A.M., *Classical and new inequalities in analysis*, Mathematics and its Applications (East European Series), vol. 61, Kluwer Academic Publishers Group, Dordrecht, 1993.

[3] Pečarić J.E., Proschan F., Tong Y.L., *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, vol. 187, Academic Press, Boston, 1992.

[4] Wang J., Deng J., Fečkan M., *Hermite-Hadamard-type inequalities for r-convex functions based on the use of Riemann-Liouville fractional integrals*, Ukrainian Math. J. **65** (2013), no. 2, 193–211.

Laboratoire des télécommunications
Faculté des Sciences et de la Technologie
University of 8 May 1945 Guelma
P.O. Box 401
24000 Guelma
Algeria
e-mail: badrimeftah@yahoo.fr