ADDITION FORMULAE WITH SINGULARITIES

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Abstract. We deal with functional equations of the form

$$f(x+y) = F(f(x), f(y))$$

(so called addition formulas) assuming that the given binary operation F is associative but its domain of definition is disconnected (admits "singularities"). The function

$$F(u,v) := \frac{u+v}{1+uv}$$

serves here as a good example; the corresponding equation characterizes the hyperpolic tangent. Our considerations may be viewed as counterparts of L. Losonczi's [4] and K. Domańska's [2] results on local solutions of the functional equation

$$f(F(x,y)) = f(x) + f(y)$$

with the same behaviour of the given associative operation F.

Our results exhibit a crucial role of 1 that turns out to be the critical value towards the range of the unknown function. What concerns the domain we admit fairly general structures (groupoids, groups, 2-divisible groups). In the case where the domain forms a group admitting subgroups of index 2 the family of solutions enlarges considerably.

1. Motivation

Although, surprisedly for some students, the *tan* function fails to be additive (i.e. in general, the equality $\tan(x+y) = \tan x + \tan y$ is invalid), we may try to save the situation by a suitable change of our understanding of addition. Namely, after setting

$$u \uplus v := \frac{u+v}{1-uv}$$

we actually have

$$\tan(x+y) = \tan x \uplus \tan y.$$

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The "new" addition $\uplus$ is not that bad: it turns out to be associative as well as commutative. These nice properties will remain unchanged if we put

$$u \oplus v := \frac{u+v}{1+uv},$$

but then we have to replace the *tan* function by its hyperbolic counterpart:

$$x \longmapsto \tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Indeed, a straightforward verification shows that

$$\tanh(x+y) = \tanh x \oplus \tanh y.$$

Now, a natural question arises: is this the only $(+, \oplus)$-additive function? More generally, given an associative binary operation F defined on a subset (possibly disconnected) of the real plane $\mathbb{R}^2$ we look for a method of solving Cauchy type functional equation of the form

$$f(x+y) = F(f(x), f(y))$$

(so called addition formulas). We admit "singularities" of the binary operation F, e.g. zeros of the denominator in the case where F is a rational two-place real-valued function.

We were inspired by the results of L. Losonczi [4] on local solutions of the functional equation

$$f(F(x,y)) = f(x) + f(y)$$

with the same behaviour of the given associative operation F. Losonczi's work was followed in a similar spirit by a paper of K. Domańska [2].

Our approach will be visualized by dealing with the case where

$$F(u,v) := u \oplus v, \qquad u, v \in \mathbb{R},\ uv \neq -1,$$

but the method applied might serve also while solving addition formulas with the corresponding binary operations

$$\begin{aligned}
F(u,v) &:= u \uplus v, && u, v \in \mathbb{R},\ uv \neq 1,\\
F(u,v) &:= \frac{1+uv}{u+v}, && u, v \in \mathbb{R},\ u+v \neq 0,\\
F(u,v) &:= \frac{uv-1}{u+v}, && u, v \in \mathbb{R},\ u+v \neq 0,\\
F(u,v) &:= \frac{u+v+2uv}{1-uv}, && u, v \in \mathbb{R},\ uv \neq 1,
\end{aligned}$$

and others.

2. Preliminaries

In the sequel, by a *groupoid* we understand an ordered pair $(G, *)$ where G is a nonempty set and $*: G \times G \longrightarrow G$ stands for an arbitrary binary operation. A function $A: G \longrightarrow \mathbb{R}$ is then called a *homomorphism* if and only if

$$A(x * y) = A(x) + A(y)$$

for all $x, y \in G$.

In the case where $*$ yields a group (semigroup) operation written additively ($* = +$) the corresponding homomorphisms will alternatively be termed *additive* functions.

Given a subgroup $(\Gamma, +)$ of a group $(G, +)$ by an *index* of Γ with respect to G we mean the cardinality of all left (right) cosets determined by Γ. The following lemma will prove to be useful in the sequel.

PROPOSITION 1. *Let $(\Gamma, +)$ be a subgroup of index 2 of a group $(G, +)$ and let $A: \Gamma \longrightarrow \mathbb{R}$ be an additive function. Then A admits an extension to an additive function mapping $(G, +)$ into the additive group $(\mathbb{R}, +)$ if and only if there exists an $a \in \Gamma' := G \setminus \Gamma$ such that*

$$\text{(a)} \qquad A(x + a) = A(a + x) \qquad \textit{for all } x \in \Gamma'.$$

The extension, whenever exists, is unique.

PROOF. The "only if" part is trivial. Let $B: G \longrightarrow \mathbb{R}$ be defined by the formula

$$B(x) := \begin{cases} A(x) & \text{for } x \in \Gamma \\ A(x + a) - \frac{1}{2}A(2a) & \text{for } x \in \Gamma' \end{cases}$$

Evidently, B is a well defined extension of A onto G. To prove that B is additive we have to fix a pair $(x, y) \in G^2$ and to distinguish the following three cases:

- $x, y \in \Gamma'$; then $x + y$ is in Γ and hence

$$\begin{aligned} B(x) + B(y) &= A(x + a) + A(y + a) - A(2a) \\ &= A(a + x) + A(y - a) = A(a + (x + y - a)) = A(x + y) \\ &= B(x + y). \end{aligned}$$

- $x \in \Gamma'$, $y \in \Gamma$; then $x + y$ is in Γ' and

$$\begin{aligned} B(x) + B(y) &= A(x + a) - \frac{1}{2}A(2a) + A(y) = A(a + x) - \frac{1}{2}A(2a) + A(y) \\ &= A(a + x + y) - \frac{1}{2}A(2a) = A(x + y + a) - \frac{1}{2}A(2a) \\ &= B(x + y). \end{aligned}$$

- $x \in \Gamma$, $y \in \Gamma'$; then $x + y$ is in Γ' and

$$B(x) + B(y) = A(x) + A(y + a) - \frac{1}{2}A(2a) = A(x + y + a) - \frac{1}{2}A(2a) \\ = B(x + y).$$

To complete the proof, assume that $\tilde{B}$ yields another additive extension of A onto G and fix arbitrarily an $x \in \Gamma'$. Then

$$2\tilde{B}(x) = \tilde{B}(2x) = A(2x) = B(2x) = 2B(x),$$

whence $\tilde{B}(x) = B(x)$ proving the uniqueness of the extension in question. □

Note that the target group $(\mathbb{R}, +)$ above may be replaced (without any change in the proof) by an arbitrary Abelian group that is uniquely 2-divisible. Extension theorem of that kind are to be found, for instance, in M. Kuczma's book [3, Chapter XVIII, §4] and in a paper of K. Dankiewicz & Z. Moszner [1], but none of them covers Proposition 1.

Another observation seems to be noteworthy: assumption (a) is trivially satisfied in Abelian groups but, as a matter of fact, (a) follows also from a considerably weaker requirement that the complement of Γ intersects the centrum of the group $(G, +)$, i.e. that there exists an element of Γ' that commutes with each member of G.

3. Singularities omitted

The simplest way to avoid difficulties caused by the disconnectedness of the domain of the binary operation considered is to restrict suitably the range of the unknown function. The risk is then that we might eliminate a number of potential solutions. Our first result of that kind reads as follows:

THEOREM 1. *Let $(G, *)$ be an arbitrary groupoid. A function $f: G \longrightarrow (-1, 1)$ yields a solution to the functional equation*

$$f(x * y) = \frac{f(x) + f(y)}{1 + f(x)f(y)} \tag{1}$$

for all $x, y \in G$, if and only if there exists a homomorphism $A: G \longrightarrow \mathbb{R}$ such that

$$f(x) = \tanh A(x), \qquad x \in G.$$

PROOF. Let $f: G \longrightarrow (-1, 1)$ be a solution of (1). Then

$$\frac{1 + f(x * y)}{1 - f(x * y)} = \left(1 + \frac{f(x) + f(y)}{1 + f(x)f(y)}\right)\left(1 - \frac{f(x) + f(y)}{1 + f(x)f(y)}\right)^{-1} \\ = \frac{1 + f(x)}{1 - f(x)} \cdot \frac{1 + f(y)}{1 - f(y)} \in (0, \infty)$$

for all $x, y \in G$. Hence

$$\frac{1}{2}\log\frac{1+f(x*y)}{1-f(x*y)} = \frac{1}{2}\log\frac{1+f(x)}{1-f(x)} + \frac{1}{2}\log\frac{1+f(y)}{1-f(y)}, \qquad x,y\in G,$$

i.e.

$$\operatorname{arctanh} f(x*y) = \operatorname{arctanh} f(x) + \operatorname{arctanh} f(y), \qquad x,y\in G,$$

which states that the function $A := \operatorname{arctanh}\circ f$ yields a homomorphism and $f = \tanh\circ A$, as claimed.

A simple calculation shows that for every homomorphism $A\colon G \longrightarrow \mathbb{R}$ the superposition $f = \tanh\circ A$, gives a solution to (1). This completes the proof. □

REMARK 1. Under an additional assumption that the unknown function $f\colon \mathbb{R} \longrightarrow (-1,1)$ is *bijective* the assertion of Theorem 1 (in the case where $(G,*) = (\mathbb{R},+)$) may be derived from Theorem 3 proved by L. Losonczi in [4]. Indeed, since (1) is then equivalent to

$$f^{-1}(u) + f^{-1}(v) = f^{-1}\left(\frac{u+v}{1-uv}\right), \qquad u,v\in(-1,1),$$

the result just quoted states that there exists an additive function $B\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f^{-1}(u) = B(\operatorname{arctanh} u), \qquad u\in(-1,1).$$

Clearly, if that is the case, B has to be a bijection of $\mathbb{R}$ onto $\mathbb{R}$, the function $A := B^{-1}$ is additive as well and $f = \tanh\circ A$.

The right hand side of equation (1) has no singularities (zeros of the denominator) also in the case where the range of the unknown function does not meet the compact interval $[-1,1]$. But then

THEOREM 2. *Let $(G,*)$ be an arbitrary groupoid. Equation* (1) *has no solutions $f\colon G \longrightarrow \mathbb{R}$ with the range $f(G)$ contained in the set $(-\infty,-1)\cup(1,\infty)$.*

PROOF. Assume the contrary: there exists a function $f\colon G \to (-\infty,-1)\cup(1,+\infty)$ that satisfies equation (1). Fix arbitrarily elements $x,y\in G$ and let us distinguish three cases:

- $f(x), f(y) \in (-\infty,-1)$,
- $f(x), f(y) \in (1,+\infty)$,
- $f(x) \in (-\infty,-1)$ whereas $f(y)\in(1,+\infty)$ (or conversely).

A straightforward verification shows that in all these possible cases the quotient

$$\frac{f(x)+f(y)}{1+f(x)f(y)}$$

falls into the interval $(-1,1)$ and, consequently, fails to be equal to $f(x*y)\in(-\infty,-1)\cup(1,+\infty)$. This contradiction finishes the proof. □

In the case where the groupoid $(G,*)$ in question is 2-*divisible*, i.e. each element $x \in G$ admits an element $y \in G$ (not necessarily unique) such that $x = y * y$, we get a considerably stronger assertion.

PROPOSITION 2. *Let $(G,*)$ be a 2-divisible groupoid. Then the range $f(G)$ of any solution $f\colon G \longrightarrow \mathbb{R}$ of equation* (1) *is contained in the interval* $[-1,1]$.

PROOF. Fix arbitrarily an $x \in G$ and take a $y \in G$ to have $x = y * y$. Then, by (1), we get

$$f(x) = f(y*y) = \frac{2f(y)}{1+f(y)^2} \in [-1,1],$$

which was to be shown. □

REMARK 2. The right hand side of equation (1) evaluated for constant functions $f(x) = c$, $x \in G$, where c is a fixed real number leads always to a nonsingular expression $c = 2c(1+c^2)^{-1}$. An immediate calculation shows that the only constant solution of (1) are $f=-1$, $f=0$ and $f=1$.

4. Singularities admissible

Definitely much more interesting seems to be the case where by a solution of equation (1) we understand *any* function $f\colon G \longrightarrow \mathbb{R}$ (with no restrictions on its range) that satisfies equality (1) for every pair $(x,y) \in G^2$ such that $f(x)f(y) \neq -1$. From now on, if not explicitely stated otherwise, we deal with solutions understood in that sense.

In non-2-divisible groupoids the assertion of Proposition 2 is invalid, in general. Indeed, consider, for example, the additive group $(\mathbb{Z},+)$ and put

$$f(1) := 2, \qquad f(n+1) := \frac{f(n)+2}{1+2f(n)}, \quad n \in \mathbb{N},$$

and

$$f(0) := 0, \qquad f(-n) := -f(n), \quad n \in \mathbb{N}.$$

A standard induction proof shows that such a function $f\colon \mathbb{Z} \longrightarrow \mathbb{R}$ yields a solution of the equation

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)} \tag{1+}$$

and, evidently, the inclusion $f(\mathbb{Z}) \subset [-1,1]$ fails to hold (actually, one may show that the set $f(\mathbb{Z}) \cap (1,\infty)$ is infinite). Noteworthy is the fact that $1 \notin f(\mathbb{Z})$. As we shall see later on, in such a case, the crucial feature of the domain considered is the existence of a subgroup of index 2 in $(\mathbb{Z},+)$, consisting of all even integers. Before stating a corresponding general result we need some preparatory lemmas.

LEMMA 1. *Let $(G,+)$ be a group. Each solution $f\colon G \longrightarrow \mathbb{R}$ of equation (1+) such that $f(0)=0$ yields an odd function.*

PROOF. On setting $y=-x$ in (1+), we see that for an arbitrarily fixed $x \in G$ either $f(x)f(-x)=-1$ or $f(-x)=-f(x)$. Thus, to prove the oddness of f it suffices to show that

$$f(x)f(-x)=-1 \quad \text{implies} \quad f(-x)=-f(x).$$

So, assume that for some $a \in G$

$$f(a)f(-a)=-1, \tag{2}$$

and let us distinguish two cases:

1. $f(2a)=f(a)$. Then $f(a)=2f(a)(1+f(a)^2)^{-1}$ which jointly with (2) forces $f(-a)$ to coincide with $-f(a)$.
2. $f(2a)\neq f(a)$. Then, setting $y=-a$ in (1+), on account of (2), we derive the relationship

$$f(x-a)=\frac{f(x)+f(-a)}{1+f(x)f(-a)}=\frac{f(x)f(a)-1}{f(a)-f(x)}$$

provided that $f(x)\neq f(a)$. With the aid of this equality applied for $x=2a$ it is a straightforward matter to show that $f(a)^2=1$ which by means of (2) implies that $f(-a)=-f(a)$

and ends the proof. □

LEMMA 2. *Let $(G,*)$ be a groupoid and let $f\colon G \longrightarrow \mathbb{R}$ be a nonconstant solution of equation* (1). *If the set*

$$S:=\{x\in G:\ f(x)=1\} \tag{3}$$

is nonempty, then the pair $(S,*|_{S\times S})$ yields a subgroupoid of $(G,*)$. If, moreover, $*$ forms a group operation then either $f(G)=\{-1,1\}$ or f is odd and

$$S\cup(s-S)=G \qquad \textit{for every } s\in S. \tag{4}$$

PROOF. For every two points $x,y\in S$ the product $f(x)f(y)=1$ is different from -1 whence $f(x*y)=(f(x)+f(y)(1+f(x)f(y))^{-1}=1$, i.e. $S*S\subset S$.

Now, assume that $*=+$ is a group operation with 0 standing for its neutral element. Setting $y=0$ in (1+) we observe first that with $c:=f(0)$ one has

$$cf(x)\neq -1 \quad \text{implies} \quad c[f(x)^2-1]=0$$

for all $x\in G$. If we had $c\neq 0$ (which implies, by putting $x=y=0$ in (1+), that $c\in\{-1,1\}$), then we would obtain the inclusion

$$f(G)\subset\left\{-1,1,-\frac{1}{c}\right\}=\{-1,1\}.$$

Because of the nonconstancy of f we have then

$$f(G) = \{-1, 1\}.$$

Therefore, in the sequel, we may assume that $c = 0$. With the aid of Lemma 1 we derive the oddness of f. To prove (4), fix an $s \in S$ to observe that for every $x \in G$ we have then either $f(x) = -1$ i.e. $x \in -S$ or, otherwise, by means of (1+) applied for $y := s$ the equality $f(x+s) = 1$ holds true, i.e. $x \in S - s$. Therefore, we conclude that (4) holds true and the proof has been completed. □

LEMMA 3. *Let $(G,+)$ be a group and let $f\colon G \longrightarrow \mathbb{R}$ be a solution of equation* (1+) *such that $f(G) = \{-1, 1\}$. Then both pairs $(S, +|_{S\times S})$ and $(S', +|_{S'\times S'})$, where S is defined by* (3) *and $S' := \{x \in G : \ f(x) = -1\}$, yield disjoint subsemigroups of the group $(G,+)$ and $S \cup S' = G$.*

Conversely, for any two disjoint subsemigroups $(S_1, +|_{S_1\times S_1})$ and $(S_{-1}, +|_{S_{-1}\times S_{-1}})$ of the group $(G,+)$ the function

$$f(x) := \begin{cases} 1 & \text{for } x \in S_1 \\ -1 & \text{for } x \in S_{-1} \end{cases}$$

is a solution of equation (1+).

PROOF. In the light of Lemma 2, only the converse part of the assertion requires a motivation. So, fix arbitrarily elements x, y in G. If both of them fall into S_1 (resp. S_{-1}) then so does $x + y$ and both sides of equation (1+) are equal to 1 (resp. -1). If $x \in S_1$ and $y \in S_{-1}$ (or conversely), then $f(x)f(y) = -1$ and there is nothing to check. This ends the proof. □

REMARK 3. The simplest possible splitting of a given group $(G,+)$ into two disjoint subsemigroups reads as follows. Let $(S,+)$ be a subsemigroup of $(G,+)$ such that $S \cap (-S) = \emptyset$, $S_1 := S$ and $S_2 := -S \cup \{0\}$. However, in some cases one of the semigroups spoken of may be considerably larger than the other. To visualize this, take a discontinuous and noninvertible additive function $A\colon \mathbb{R} \longrightarrow \mathbb{R}$ and put

$$S_1 := \{x \in \mathbb{R} : \ A(x) > 0\} \quad \text{and} \quad S_2 := \{x \in \mathbb{R} : \ A(x) \le 0\}.$$

Clearly, we obtain the desired splitting of the additive group $(\mathbb{R},+)$ but S_2 is not just the reflection (up to the neutral element) of S_1 with respect to 0 but up to the kernel of A which may be pretty large indeed (see e.g M. Kuczma [3]).

Now, we proceed with a description of solutions of (1+) vanishing at zero and having 1 off their ranges.

THEOREM 3. *Let $(G,+)$ be a group (not necessarily commutative) and let $f\colon G \longrightarrow \mathbb{R}$ be a solution of equation* (1+) *such that $f(0) = 0$ and*

$$(*) \qquad 1 \notin f(G) \not\subset (-1, 1).$$

Then there exists a subgroup $(\Gamma,+)$ of the group $(G,+)$ of index 2, and an additive function $B\colon G \longrightarrow \mathbb{R}$ such that $\ker B \subset \Gamma$ *and*

$$f(x) = \begin{cases} \tanh \circ B(x) & \textit{for } x \in \Gamma \\ \coth \circ B(x) & \textit{for } x \in G \setminus \Gamma. \end{cases}$$

Conversely, any function f of that form yields a solution to equation (1+) *and satisfies condition* $(*)$.

PROOF. Let $x_0 \in G$ be such that $|f(x_0)| > 1$. On account of Lemma 1, f is necessarily odd which implies that

$$\Gamma := \{x \in G : \ f(x) \in (-1,1)\} = -\Gamma.$$

Clearly, $\Gamma \neq \emptyset$ because 0 is in Γ (actually, Γ contains also a nonzero element $2x_0$). To show that the pair $(\Gamma, +|_{\Gamma\times\Gamma})$ forms a subgroup of $(G,+)$ it suffices now to check that

$$\Gamma + \Gamma \subset \Gamma. \tag{5}$$

To this end, fix $x, y \in \Gamma$ and note that then $|f(x)f(y)| < 1$ forcing $f(x)f(y)$ to be different from -1. By (1+) and a simple calculation (omitted) we infer that

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)} \in (-1,1)$$

which proves that $x+y \in \Gamma$. Thus $f|_\Gamma$ yields a solution to (1+) on a group $(\Gamma,+)$, whose range is contained in the open interval $(-1,1)$. An appeal to Theorem 1 quarantees the existence of an additive map $A\colon \Gamma \longrightarrow \mathbb{R}$ such that

$$f(x) = \tanh \circ A(x) \qquad \text{for all } x \in \Gamma.$$

Now, we are going to show that the complement $\Gamma' := G \setminus \Gamma$ of Γ enjoys the property

$$\Gamma' + \Gamma' \subset \Gamma. \tag{6}$$

Indeed, taking arbitrary x, y from Γ' by means of the assumption we have to have $|f(x)f(y)| > 1$ which allows to apply equation (1+) to get

$$|f(x+y)| = \left|\frac{f(x)+f(y)}{1+f(x)f(y)}\right| < 1,$$

(cf. the proof of Theorem 2).

Since Γ' is nonvoid (recall that $x_0 \in \Gamma'$), taking (5) and (6) into account, we conclude that $(\Gamma,+)$ is a subgroup of index 2 in $(G,+)$, as claimed. In particular, there exists an element $a \in \Gamma'$ such that

$$\Gamma' = \Gamma - a \quad \text{and} \quad \Gamma \cup (\Gamma - a) = G.$$

It remains to determine the values of f on Γ'. To this end, fix arbitrarily an $x \in \Gamma'$. Then $x + a \in \Gamma$ and $|f(x)f(a)| > 1$, whence

$$u := \tanh A(x+a) = f(x+a) = \frac{f(x)+f(a)}{1+f(x)f(a)}.$$

Consequently,

$$f(x)\,(uf(a)-1) = f(a) - u \neq 0$$

because $|u| < 1 < |f(a)|$. Thus

$$f(x) = \frac{\tanh \circ A(x+a) - f(a)}{1 - f(a)\tanh \circ A(x+a)}$$

and since $|f(a)| > 1$ there exists a uniquely determined real number α such that $f(a) = \coth \alpha$. An elementary calculation shows now that the equality

$$f(x) = \coth(A(x+a) - \alpha)$$

holds true for all x from Γ'. In particular, since a also belongs to Γ',

$$\coth \alpha = f(a) = \coth(A(2a) - \alpha)$$

which by the invertibility of the *coth* function implies that $\alpha = \frac{1}{2}A(2a)$. Moreover, for every $x \in \Gamma'$ we have

$$\begin{aligned}\tanh A(a+x) &= f(a+x) = \frac{f(a)+f(x)}{1+f(a)f(x)} = \frac{f(x)+f(a)}{1+f(x)f(a)} = f(x+a)\\ &= \tanh A(x+a).\end{aligned}$$

Thus $A(a+x) = A(x+a)$ for all x from Γ' whence, by Proposition 1, there exists a unique additive extension $B\colon G \longrightarrow \mathbb{R}$ of the function A. Clearly,

$$A(x+a) - \frac{1}{2}A(2a) = B(x+a) - \frac{1}{2}B(2a) = B(x)$$

for every $x \in \Gamma'$ and, finally,

$$f(x) = \begin{cases} \tanh \circ B(x) & \text{for } x \in \Gamma \\ \coth \circ B(x) & \text{for } x \in G \setminus \Gamma, \end{cases}$$

as claimed. This formula implies also that the kernel of B and the complement of Γ have to be disjoint, i.e. that $\ker B \subset \Gamma$.

We omit here somewhat tedious but quite elementary calculations based on the fact that relationships (5) and (6) are both satisfied and showing that for an arbitrary subgroup $(\Gamma, +)$ of index 2 in $(G, +)$, an arbitrary additive function $B\colon G \longrightarrow \mathbb{R}$ such that $\ker B \subset \Gamma$ the above formula establishes a solution of equation (1+) and satisfies condition $(*)$.

Thus the proof has been completed. □

COROLLARY 1. *If a given group $(G, +)$ does not admit subgroups of index 2, then the only real solutions of equation* (1+) *having* 1 *off their ranges are the canonical ones:* $\tanh \circ A$ *where $A\colon G \longrightarrow \mathbb{R}$ is an arbitrary additive function.*

REMARK 4. The example exhibited at the beginning of the present section refers to the case where $(G,+) = (\mathbb{Z},+)$, $\Gamma = 2\mathbb{Z}$, $A(x) = \left(\frac{1}{2}\log 3\right)\cdot x$, $x \in \Gamma$ and

$$f(x) = \begin{cases} \tanh\left(\left(\frac{1}{2}\log 3\right)\cdot x\right) & \text{for } x \in 2\mathbb{Z} \\ \coth\left(\left(\frac{1}{2}\log 3\right)\cdot x\right) & \text{for } x \in 2\mathbb{Z}+1 \end{cases}$$

This observation may also be viewed as a solution of the recurrence relation defining the function $f\colon \mathbb{Z} \longrightarrow \mathbb{R}$ spoken of right before Lemma 1.

Finally, we will examine the case where the critical value 1 falls into the range of an unknown function. Jointly with Theorems 1, 3 and Lemma 3 this will give a complete description of *all* real solutions of equation (1+) defined on arbitrary groups.

To simplify the notation, in what follows we shall disregard visualizing the necessary restrictions of the binary operation in question to substructures of the group considered.

THEOREM 4. *Let $(G,+)$ be a group (not necessarily commutative) and let $f\colon G \longrightarrow \mathbb{R}$ be a nonconstant solution of equation (1+) vanishing at zero and such that the set $S := f^{-1}(\{1\})$ is nonvoid. Then the pair $(S,+)$ forms a subsemigroup of $(G,+)$, $S\cap(-S) = \emptyset \neq G_0 := G\setminus(S\cup(-S))$, $(G_0,+)$ is a subgroup of $(G,+)$ and*

$$S + G_0 = S \quad \textit{as well as} \quad -S + G_0 = -S, \tag{7}$$

whereas the function $f_0 := f|_{G_0}$ yields a solution of equation (1+) on the group $(G_0,+)$ enjoying the property $1 \notin f_0(G_0)$.

Conversely, let $(S,+)$ be an arbitrary subsemigroup of $(G,+)$ such that $S\cap(-S) = \emptyset \neq G_0 := G\setminus(S\cup(-S))$, $(G_0,+)$ is a subgroup of $(G,+)$ and relationships (7) hold true. Then, for any solution $f_0\colon G_0 \longrightarrow \mathbb{R}$ of (1+) having 1 off its range the function $f\colon G \longrightarrow \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \textit{for } x \in S \\ -1 & \textit{for } x \in -S \\ f_0(x) & \textit{for } x \in G_0 \end{cases} \tag{8}$$

satisfies equation (1+) on G.

PROOF. *Necessity.* The oddness of f as well as the semigroup structure of $S := f^{-1}(\{1\})$ result from Lemma 1 and 2, respectively. Clearly, now $-S = f^{-1}(\{-1\})$ and S are disjoint and $0 \in G_0 := G\setminus(S\cup(-S))$. We shall first show that equalities (7) are fulfilled. In fact, $S = S+0 \subset S+G_0$ and if $x \in S+G_0$ then $x = s+a$ with $f(s) = 1$ and $f(a) \notin \{-1,1\}$ whence $f(s)f(a) = f(a) \neq -1$ and, by (1+), we get $f(x) = f(s+a) = 1$, i.e. $x \in S$. The proof of the second equality in (7) is literally the same.

To show that $(G_0, +)$ is a subgroup of $(G, +)$ observe that $G_0 = -G_0$ and it suffices to check that

$$G_0 + G_0 \subset G_0.$$

To this end, fix arbitrarily two elements x, y from G_0 and suppose, for the indirect proof, the sum $x + y$ to be off G_0. Then $x + y \in S$ or $x + y \in -S$. In the first case, since $-y \in G_0$, in view of (7) we would get $S \ni x = x + y + (-y) \in S$, a contradiction. In the other case we proceed quite analogously.

By the definition of G_0 the value 1 (as well as -1) lies off the range of the function $f|_{G_0}$.

Sufficiency. To prove that formula (8) establishes a solution to (1+) we have to fix a pair $(x, y) \in G^2$ and to distinguish the following cases:

(i) $x, y \in S$	(iv) $x \in S,\ y \in G_0$
(ii) $x, y \in -S$	(v) $x \in -S,\ y \in G_0$
(iii) $x \in S,\ y \in -S$	(vi) $x, y \in G_0$.

Ad (i) & (ii). So does $x + y$ and both sides of (1+) are equal to 1 (resp. -1).

Ad (iii). Then $f(x)f(y) = -1$ and there is nothing to check.

Ad (iv) & (v). Then $f(x)f(y) = \pm f(y) \neq -1$ and, by (7), $x+y \in \pm S$ and both sides of (1+) are equal to ± 1.

Ad (vi). So does $x + y$ and and we make use of the fact that f_0 is a solution of (1+) on G_0.

This ends the proof. □

We terminate this paper with some examples.

5. Three examples

With the aid of Theorem 4 one may produce numerous nontrivial solutions of equation (1+) admitting the value 1 in their ranges. By way of illustration let us consider the following two situations.

1. Take, like in Remark 3, a discontinuous and noninvertible additive function $A\colon \mathbb{R} \longrightarrow \mathbb{R}$ and put $S := \{x \in \mathbb{R} : A(x) > 0\}$. Then $(S, +)$ yields a subsemigroup of $(G, +)$ and $-S = \{x \in \mathbb{R} : A(x) < 0\}$ does not intersect S. Further, $G_0 := G \setminus (S \cup (-S)) = \ker A \neq \emptyset$; as a matter of fact, G_0 is uncountable and, obviously, $(G_0, +)$ forms a subgroup of the additive group $(\mathbb{R}, +)$. Moreover, $x \in S + G_0$ implies that $x = s + a$ where $A(s) > 0$ and $A(a) = 0$ and due to the additivity of A we have $A(x) = A(s) > 0$ which states that $x \in S$. In other words: $S + G_0 \subset S$. The converse inclusion is trivial because 0 is in G_0. Similarly we prove that $-S = -S + G_0$ getting equalities (7).

Now, setting

$$f(x) := \begin{cases} 1 & \text{for } x \in S \\ -1 & \text{for } x \in -S \\ \tanh x & \text{for } x \in \ker A \end{cases}$$

we obtain a solution to (1+). Plainly, the identity function under the sign of tanh function may be replaced by an arbitrary additive function $A_0 \colon \ker A \longrightarrow \mathbb{R}$.

2. Let $(\mathbb{Z}[\sqrt{2}\,], +)$ stand for the additive group of the simple extension of the ring of all integers by $\sqrt{2}$. Take

$$S := \{p + q\sqrt{2} \in \mathbb{Z}[\sqrt{2}\,] : \ q > 0\}.$$

Then $(S, +)$ forms a semigroup, $-S = \{p + q\sqrt{2} \in \mathbb{Z}[\sqrt{2}\,] : \ q < 0\}$ and S are disjoint and $G_0 := \mathbb{Z}[\sqrt{2}\,] \setminus (S \cup (-S))$ coincides with $\mathbb{Z}$. It is also easily seen that

$$S + \mathbb{Z} = S \quad \text{as well as} \quad -S + \mathbb{Z} = -S.$$

Fix arbitrarily an $\alpha \in \mathbb{R} \setminus \{0\}$ and put

$$f(x) := \begin{cases} 1 & \text{for } x \in S \\ -1 & \text{for } x \in -S \\ \tanh \alpha x & \text{for } x \in 2\mathbb{Z} \\ \coth \alpha x & \text{for } x \in 2\mathbb{Z} + 1. \end{cases}$$

By means of Theorems 4 and 3 the function f provides a solution of equation (1+) on the group $(\mathbb{Z}[\sqrt{2}\,], +)$ assuming the values 1 and -1 as well as infinitely many values off the interval $[-1, 1]$.

3. The role of the hypothesis that $\ker B \subset \Gamma$ occurring in the statement of Theorem 3 may readably be observed while considering the multiplicative group $(\mathbb{R}^*, \cdot)$ of all nonzero real numbers. This group possesses a subgroup of index 2, namely $(\Gamma, \cdot) = ((0, \infty), \cdot)$. It is well known (see M. Kuczma [3], for example) that each homomorphism $B \colon \mathbb{R}^* \longrightarrow \mathbb{R}$ is of the form $B(x) = a(\log |x|)$, $x \in \mathbb{R}^*$, where $a \colon \mathbb{R} \longrightarrow \mathbb{R}$ stands for an additive function. Thus $-1 \in \ker B$ whence

$$\ker B \not\subset \Gamma = (0, \infty).$$

In the light of Theorem 3, even in the most regular case $A(x) = x$, $x \in \mathbb{R}$, there is no way to extend the corresponding function

$$f(x) := \begin{cases} \dfrac{x^2 - 1}{x^2 + 1} & \text{for } x \in (0, \infty) \\[2ex] \dfrac{x^2 + 1}{x^2 - 1} & \text{for } x \in (-\infty, 0) \setminus \{-1\} \end{cases}$$

to a solution of the equation

$$f(xy) = \frac{f(x) + f(y)}{1 + f(x)f(y)}$$

to be valid for all $x, y \in \mathbb{R}^*$.

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