SUPERSTABILITY OF THE D'ALEMBERT FUNCTIONAL EQUATION IN L_p^+ SPACES

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Abstract. Let $(X, +, -, 0, \Sigma, \mu)$ be an abelian complete measurable group with $\mu(X) > 0$. Let $f: X \longrightarrow \mathbb{C}$ be a function. We will show that if $A(f) \in L_p^+(X \times X, \mathbb{C})$ where

$$A(f)(x, y) = f(x + y) + f(x - y) - 2f(x)f(y), \qquad x, y \in X,$$

then $f \in L_p^+(X, \mathbb{C})$ or there exists exactly one function $g: X \longrightarrow \mathbb{C}$ with

$$g(x + y) + g(x - y) = 2g(x)g(y), \qquad x, y \in X$$

such that f is equal to g almost everywhere with respect to the measure μ.

L_p^+ denotes the space of all functions for which the upper integral of $\|f\|^p$ is finite.

1. Introduction

DEFINITION 1. The functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y) \tag{1}$$

is known as the d'Alembert functional equation (see [1], [4]).

A standard symbol $\mathbb{C}$ denotes the set of complex numbers, for a set X a symbol $\mathbb{C}^X$ denotes a set of all functions $f: X \longrightarrow \mathbb{C}$.

DEFINITION 2. Let X be an abelian semigroup. The d'Alembert difference operator $A: \mathbb{C}^X \longrightarrow \mathbb{C}^{X^2}$ is defined by

$$A(f)(x, y) := f(x + y) + f(x - y) - 2f(x)f(y), \qquad x, y \in X. \tag{2}$$

Let $f: X \longrightarrow \mathbb{C}$, where X is an abelian semigroup. We will consider the following problem of stability. Let us suppose that $A(f)$ is bounded in a certain sense. What does it imply? In the case of the d'Alembert functional equation the phenomen of

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superstability occurs which means that either f is bounded in the same sense as $A(f)$ or f satisfies the d'Alembert functional equation.

Boundness in different senses can be considered. The first result of this type for the d'Alembert functional equation was obtained by Baker in [3] (see also [7]). He has proved the following theorem.

THEOREM 1 ([3], [7]). *Let $\delta > 0$ and G be an abelian group and $f: G \longrightarrow \mathbb{C}$ be a function satisfying the inequality*

$$\forall x, y \in G \quad |A(f)(x,y)| \leq \delta.$$

Then either f is bounded or satisfies the d'Alembert functional equation (1).

In the present paper we will consider the stability in a generalization of L^p spaces – we will prove that if $A(f) \in L_p^+(X \times X, \mathbb{C})$ (p-power of the modulus of $A(f)$ is bounded by an integrable function) then $f \in L_p^+(X, \mathbb{C})$ or f satisfies the d'Alembert functional equation (1) almost everywhere. In this case we call such stability "almost superstability".

We shall show under some additional assumptions that if $A(f)(x,y) = 0$ almost everywhere then there exists exactly one function $g: X \longrightarrow \mathbb{C}$ satisfying the d'Alembert functional equation (1) such that f is equal to g almost everywhere.

For the Cauchy functional equation

$$f(x+y) = f(x) + f(y)$$

similar problem has been investigated by Józef Tabor in [8] (see also [4], [6]).

For the equation of quadratic functionals

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

superstability in L_p^+ spaces was considered by Stefan Czerwik and Krzysztof Dłutek in [5] (see also [4]).

2. Preliminaries

DEFINITION 3 ([8], see also [4] and [5]). We say that $(X, +, -, 0, \Sigma, \mu)$ is an abelian complete measurable group, if

(a) $(X, +, -, 0)$ is an abelian group,

(b) (X, Σ, μ) is σ-finite measure space, μ is not identically equal to zero and is complete,

(c) the σ-algebra Σ and the measure μ are invariant with respect to the left translations and μ is invariant under symmetry with respect to zero,

(d) $\nu = \mu \times \mu$ is the completion of the product measure $X \times X$,

(e) the translation $S: X \times X \longrightarrow X \times X$ defined by

(3) $$S((x,y)) = (x, x+y)$$

is measurability preserving, i.e. S and S^{-1} are measurable.

Under the assumptions given above the measure μ is invariant with respect to translations and symmetry with respect to zero.

DEFINITION 4 ([8], see also [4] and [5]). Let (X,μ) be a measure space. A symbol $L(X,\mathbb{R})$ denotes the space of all integrable functions $\varphi\colon X \longrightarrow \mathbb{R}$.

Moreover, if $f\colon X \longrightarrow \mathbb{R}$ is nonnegative we define the upper integral of f with respect to μ by

$$\int_X^+ f\,d\mu := \inf\left\{\int_X \varphi\,d\mu \mid \varphi \in L(X,\mathbb{R}),\ f(x) \le \varphi(x),\ x \in X\right\},$$

or

$$\int_X^+ f\,d\mu = +\infty$$

if the corresponding set is empty.

Let $p > 0$. Then we define the space

$$L_p^+(X,\mathbb{C}) := \left\{f\colon X \longrightarrow \mathbb{C} \mid \int_X^+ |f|^p\,d\mu < +\infty\right\}.$$

LEMMA 1 ([8], see also [4]). *Let (X,Σ,μ) be a measure space and let $p > 0$. If $f,g \in L_p^+(X,\mathbb{C})$, then $f+g \in L_p^+(X,\mathbb{C})$.*

DEFINITION 5. For any function $f\colon X \longrightarrow Y$ and $x_0 \in X$ we define

$$f_{x_0}(x) := f(x+x_0), \qquad x \in X.$$

LEMMA 2 ([8], see also [4]). *Let $(X,+,-,0,\Sigma,\mu)$ be an abelian measurable group. Let $f \in L_p^+(X,\mathbb{C})$ and $p > 0$. Then*

$$\forall\, x_0 \in X \quad f_{x_0} \in L_p^+(X,\mathbb{C}).$$

LEMMA 3 ([8], see also [4]). *Let (X,Σ,μ) be a measure space and let $p > 0$ and $f \in L_p^+(X \times X,\mathbb{C})$. Then there exists a subset $A \subset X$ such that $\mu(A) = 0$ and*

$$f(\cdot,y) \in L_p^+(X,\mathbb{C}) \qquad \textit{for } y \in X \setminus A.$$

REMARK 1.1. Obviously, there exists a subset $B \subset X$ such that $\mu(B) = 0$ and

$$f(x,\cdot) \in L_p^+(X,\mathbb{C}) \qquad \text{for } x \in X \setminus B.$$

LEMMA 4 ([5]). *Let $(X,+,-,0,\Sigma,\mu)$ be an abelian complete measurable group and let $A \subset X$, $\mu(A) = 0$. If*

$$D = \{(x,y) \in X \times X \mid x \in A \vee y \in A \vee x+y \in A \vee x-y \in A\},$$

then $\nu(D) = 0$.

3. Superstability of the d'Alembert functional equation

THEOREM 2. *Let $(X, +, -, 0, \Sigma, \mu)$ be an abelian complete measurable group and let $f\colon X \longrightarrow \mathbb{C}$ be a function such that $A(f) \in L_p^+(X \times X, \mathbb{C})$, for some $p > 0$. Then $f \in L_p^+(X, \mathbb{C})$ or*

$$\forall x, y \in X \quad f(x+y) + f(x-y) \overset{\nu}{=} 2f(x)f(y). \tag{4}$$

PROOF. Let us assume that $f \notin L_p^+(X, \mathbb{C})$, then from the definition it follows that

$$\int_X^+ |f(x)|^p \, d\mu(x) = +\infty.$$

On account of Lemma 2 there exists a subset $A \subset X$ such that $\mu(A) = 0$ and

$$\forall y \in X \setminus A \quad A(f)(\cdot, y) \in L_p^+(X, \mathbb{C}).$$

Let $u, v, x \in X$, then we obtain

$$\begin{aligned}
2f(x)[A(f)(u,v)] &= 2f(x)[f(u+v) + f(u-v) - 2f(u)f(v)] \\
&= 2f(x)f(u+v) + 2f(x)f(u-v) - 4f(x)f(u)f(v) \\
&= [f((x+u)+v) + f(x+u-v) - 2f(x+u)f(v)] \\
&\quad - [f(x+(u+v)) + f(x-u-v) - 2f(x)f(u+v)] \\
&\quad - [f(x+(u-v)) + f(x-u+v) - 2f(x)f(u-v)] \\
&\quad + [f((x-u)+v) + f(x-u-v)) - 2f(x-u)f(v)] \\
&\quad + 2f(v)[f(x+u) + f(x-u) - 2f(x)f(u)] \\
&= A(f)(x+u,v) - A(f)(x,u+v) - A(f)(x,u-v) \\
&\quad + A(f)(x-u,v) + 2f(v)A(f)(x,u) \\
&= (A(f)(x,v))_u - A(f)(x,u+v) - A(f)(x,u-v) \\
&\quad + (A(f)(x,v))_{-u} + 2f(v)A(f)(x,u).
\end{aligned}$$

Consequently, for $u, v, x \in X$ we have

$$\begin{aligned}
2f(x)[A(f)(u,v)] = (A(f)(x,v))_u &- A(f)(x,u+v) - A(f)(x,u-v) \\
&+ (A(f)(x,v))_{-u} + 2f(v)A(f)(x,u).
\end{aligned}$$

Take $u, v \in X \setminus A$ such that $u + v \in X \setminus A$ and $u - v \in X \setminus A$. In view of the previous lemmas we see that the right side of the last equality as a function of x belongs to $L_p^+(X, \mathbb{C})$, which means that

$$\int_X^+ |2f(x)[A(f)(u,v)]|^p \, d\mu(x) < +\infty,$$

and in view of the assumption that $f \notin L_p^+(X, \mathbb{C})$, hence it follows

$$A(f)(u,v) = 0 \qquad \text{for } u, v \in X \setminus A, \ u+v \in X \setminus A, \ u-v \in X \setminus A.$$

One can rewrite this condition in the form

$$A(f)(u,v) = 0 \qquad \text{for } (u,v) \in X \times X \setminus D,$$

where D is as in Lemma 2. Since by this lemma, $\nu(D) = 0$ and the proof is complete. □

THEOREM 3. *Let $(X, +, -, 0, \Sigma, \mu)$ be an abelian complete measurable group with $\mu(X) > 0$ and let $f: X \longrightarrow \mathbb{C}$ be a function such that*

$$A(f)(x,y) \stackrel{\nu}{=} 0.$$

Then there exists exactly one function $g: X \longrightarrow \mathbb{C}$ with

$$g(x+y) + g(x-y) = 2g(x)g(y)$$

such that

$$f(x) \stackrel{\mu}{=} g(x) \qquad \text{for } x \in X.$$

REMARK 3.1. A similar result with different assumptions for almost trigonometric functions was proved by I. Adamaszek in her paper [2], but we provide a different proof fixed to L_p^+ spaces.

The proof is very similar to the proof of Theorem 1 from the paper of S. Czerwik and K. Dłutek ([5]) and changes only in a few places, thus we will use their method of the proof here.

PROOF OF THEOREM 3. If $f = 0$ almost everywhere then $g = 0$ and the theorem holds. Thus let us assume that there exists a subset $A \subset X$, $\mu(A) > 0$ such that $f(x) \neq 0$ for $x \in A$. By assumption, there exists a set $V \subset X \times X$ such that $\nu(V) = 0$ and

$$\forall\, (x,y) \in X \times X \setminus V \quad f(x+y) + f(x-y) = 2f(x)f(y).$$

Thus by Fubini's theorem there exist sets $U_1, U_2 \subset X$ such that $\mu(U_1) = \mu(U_2) = 0$ and

(a) for every $x \in X \setminus U_1$ there exists $K_x \subset X$ such that $\mu(K_x) = 0$ and for all $y \in X \setminus K_x$ we have $A(f)(x,y) = 0$;

(b) for every $y \in X \setminus U_2$ there exists $L_y \subset X$ such that $\mu(L_y) = 0$ and for all $x \in X \setminus L_y$ we have $A(f)(x,y) = 0$.

Let $U := U_1 \cup U_2$. Then, obviously, $\mu(U) = 0$. For any $x \in X$, we define

$$U_x := U \cup (x - U) \cup (-x + U).$$

Clearly, $\mu(U_x) = 0$, whence $X \setminus U_x \neq \emptyset$. Consequently, for every $x \in X$ there exists $w_x \in X \setminus U_x$, i.e.

$$w_x \notin U, \quad x + w_x \notin U, \quad x - w_x \notin U,$$

and $f(w_x) \neq 0$ (it is possible by assumption that $\mu(A) > 0$, where A is defined at the beginning of the proof). Let us define the function $g: X \longrightarrow \mathbb{C}$ by the formula

$$g(x) := \frac{f(x+w_x)+f(x-w_x)}{2f(w_x)}. \tag{5}$$

First we shall show that g does not depend on the choice of $w_x \in X \setminus U_x$. Take any $x \in X$, then $x + w_x \notin U$, and $x - w_x \notin U$. Thus by (a) we get

$$2f(x+w_x)f(y) = f(x+y+w_x)+f(x-y+w_x) \tag{6}$$

for $y \in X \setminus K_{x+w_x}$, and

$$2f(x-w_x)f(y) = f(x+y-w_x)+f(x-y-w_x) \tag{7}$$

for $y \in X \setminus K_{x-w_x}$.

Analogously, in view of (b) (substituting $y = w_x$ and taking x as $x+y$ or $x-y$), we obtain

$$2f(x+y)f(w_x) = f(x+y+w_x)+f(x+y-w_x) \tag{8}$$

for $x + y \in X \setminus L_{w_x}$, i.e. $y \in X \setminus (L_{w_x} - x)$, and

$$2f(x-y)f(w_x) = f(x-y+w_x)+f(x-y-w_x) \tag{9}$$

for $x - y \in X \setminus L_{w_x}$, i.e. $y \in X \setminus (-L_{w_x} + x)$.

Let us denote

$$A_{w_x} := K_{x+w_x} \cup K_{x-w_x} \cup (L_{w_x} - x) \cup (-L_{w_x} + x).$$

Then we have $\mu(A_{w_x}) = 0$. Adding the equations (6) and (7) and then the equations (8) and (9) side by side we obtain

$$\begin{aligned} 2f(y)[f(x+w_x)+f(x-w_x)] &= f(x+y+w_x)+f(x-y+w_x) \\ &\quad + f(x+y-w_x)+f(x-y-w_x), \\ 2f(w_x)[f(x+y)+f(x-y)] &= f(x+y+w_x)+f(x+y-w_x) \\ &\quad + f(x-y+w_x)+f(x-y-w_x). \end{aligned}$$

Comparing sides we get

$$2f(y)[f(x+w_x)+f(x-w_x)] = 2f(w_x)[f(x+y)+f(x-y)],$$

and taking into account that $f(w_x) \neq 0$ finally we come to the equality

$$2f(y)g(x) = f(x+y)+f(x-y) \tag{10}$$

valid for $y \in X \setminus A_{w_x}$. We can find $y \in X \setminus A_{w_x}$ such that $f(y) \neq 0$ and then

$$g(x) = \frac{f(x+y)+f(x-y)}{2f(y)}. \tag{11}$$

Consider any two element $w_x^1, w_x^2 \in X \setminus U_x$ such that $f(w_x^1) \neq 0$ and $f(w_x^2) \neq 0$. We can find $y \in X \setminus (A_{w_x^1} \cup A_{w_x^2})$ such that $f(y) \neq 0$. Consequently by (11) we obtain

$$g_n(x) = \frac{f(x+y) + f(x-y)}{2f(y)}, \qquad n = 1, 2,$$

where g_n, $n = 1, 2$, are defined by (5) for $w_x = w_x^n$, $n = 1, 2$. Therefore,

$$g_1(x) = g_2(x) = g(x)$$

which means that g does not depend on the choice of $w_x \in X \setminus U_x$.

Now we will show that $f = g$ almost everywhere. Indeed, if $x \in X \setminus U$, we can find $w_x \in X \setminus (U_x \cup K_x)$ such that $f(w_x) \neq 0$ and hence on account of (a), we infer that

$$2f(x)f(w_x) = f(x + w_x) + f(x - w_x).$$

Consequently,

$$f(x) = \frac{f(x + w_x) + f(x - w_x)}{2f(w_x)},$$

i.e. $f(x) = g(x)$ for $X \setminus U$.

We will verify that g satisfies the d'Alembert functional equation (1). Let us notice that $\mu(U_x) = 0$ for every $x \in X$. Let $x, y \in X$ be arbitrarily fixed. Thus for $b \in X \setminus U_y$ such that $f(b) \neq 0$, on account de Morgan's law, we have

$$\begin{aligned} Z := {} & [X \setminus U_x] \cap [(X \setminus (U_{x+y} \cup U_{x-y})) - b] \cap [(X \setminus (U_{x+y} \cup U_{x-y})) + b] \\ & \cap [X \setminus L_b] \cap [(X \setminus (L_{y+b} \cup L_{y-b})) - x] \cap [X \setminus (x - (L_{y-b} \cup L_{y+b}))] \\ \neq {} & \emptyset. \end{aligned}$$

Hence, for $b \in X \setminus U_y$, there exists $a \in Z$, $f(a) \neq 0$, which by definition of Z and standard properties of algebra of sets, is equivalent to

$$\begin{array}{cc} a \in X \setminus U_x, & a + b \in X \setminus (U_{x+y} \cup U_{x-y}), \\ a - b \in X \setminus (U_{x+y} \cup U_{x-y}), & x + a \in X \setminus (L_{y+b} \cup L_{y-b}), \\ x - a \in X \setminus (L_{y-b} \cup L_{y+b}), & a \in X \setminus L_b, \\ b \in X \setminus U, & y + b \in X \setminus U, \\ y - b \in X \setminus U. & \end{array}$$

Taking into account that the definition of $g(x)$ does not depend on choosing $w_x \in X \setminus U_x$, by (10) we get

$$\begin{aligned} 2f(a)g(x) &= f(x+a) + f(x-a), \\ 2f(b)g(y) &= f(y+b) + f(y-b), \\ 2f(a+b)g(x+y) &= f(x+y+a+b) + f(x+y-a-b), \\ 2f(a-b)g(x+y) &= f(x+y+a-b) + f(x+y-a+b), \\ 2f(a-b)g(x-y) &= f(x-y+a-b) + f(x-y-a+b), \\ 2f(a+b)g(x-y) &= f(x-y+a+b) + f(x-y-a-b). \end{aligned}$$

From the above equalities, we obtain

$$\begin{aligned}4f(a)f(b)g(x)g(y) &= [f(x+a)+f(x-a)][f(y+b)+f(y-b)]\\ &= f(x+a)f(y+b)+f(x+a)f(y-b)\\ &\quad + f(x-a)f(y+b)+f(x-a)f(y-b)\\ &= \frac{1}{2}[f(x+y+a+b)+f(x-y+a-b)]\\ &\quad + \frac{1}{2}[f(x+y+a-b)+f(x-y+a+b)]\\ &\quad + \frac{1}{2}[f(x+y-a+b)+f(x-y-a-b)]\\ &\quad + \frac{1}{2}[f(x+y-a-b)+f(x-y-a+b)]\\ &= [f(a+b)+f(a-b)][g(x+y)+g(x-y)],\end{aligned}$$

thus finally we get $2g(x)g(y) = g(x+y)+g(x-y)$.

To prove the uniqueness part, assume that we have two functions $g_n\colon X \longrightarrow \mathbb{C}$, $n = 1,2$ satisfying the d'Alembert functional equation (1) and μ-equivalent to f. Then $g_1(x) = g_2(x)$ for all $x \in X \setminus B$ where $\mu(B) = 0$. For an arbitrarily fixed $x \in X$ we can find $y \in X \setminus [B \cup (B-x) \cup (x-B)]$ such that $y \notin B$, $x+y \notin B$, $x-y \notin B$ and $f(y) \neq 0$, whence

$$g_1(x) = \frac{g_1(x+y)+g_1(x-y)}{2g_1(y)} = \frac{g_2(x+y)+g_2(x-y)}{2g_2(y)} = g_2(x).$$

This concludes the proof. □

COROLLARY 1. *Let X be an abelian complete measurable group, $\mu(X) > 0$ and let $f\colon X \longrightarrow \mathbb{C}$ be a function such that $f \notin L_p^+(X,\mathbb{C})$. The following conditions are equivalent:*

(i) *$A(f) \in L_p^+(X \times X, \mathbb{C})$ for some $p > 0$;*

(ii) *there exists a function $g\colon X \longrightarrow \mathbb{C}$ satisfying the d'Alembert functional equation (1) such that $g(x) = f(x)$ almost everywhere.*

PROOF. The proof follows from the previous theorems. □

REMARK 3.2. The implication (ii) $\Rightarrow$ (i) is true for any $f\colon X \longrightarrow \mathbb{C}$ which is obvious.

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