ON THE RELATION BETWEEN THE ITÔ AND STRATONOVICH INTEGRALS IN HILBERT SPACES

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Abstract. We examine the relation between the Itô and Stratonovich integrals in Hilbert spaces. A transition formula has origin in the correction term of the Wong–Zakai approximation theorem.

1. Introduction

We prove the existence of the Stratonovich integral of a solution (in the mild sense) of a semilinear evolution equation in a Hilbert space. Thus we examine the relation between the Itô and Stratonovich integrals with respect to a Hilbert space valued Wiener process and with integrands being some nonlinear operators in another Hilbert space. The result is of interest because the process considered as the integral is not a semimatringale; more exactly, the stochastic convolution has to be considered. A transition formula contains a complementary term that is the same as the correction term in the Wong–Zakai approximation theorem with a nonlinear operator under the stochastic integral (see Twardowska [9] and [10]). Such an infinite-dimensional form of the correction term was also proved for nonlinear stochastic partial differential equations and for stochastic Navier–Stokes equations by Twardowska in [11] and [12].

As is well known, the Itô stochastic integral is convenient in some problems because it is the martingale. However, the Stratonovich integral is particularly preferable in applications. Its advantage over the Itô integral in computational techniques is that we can work within these integrals in the same way as with the ordinary integrals of smooth functions (see Stratonovich [8]). For this reason the Stratonovich integral has been discussed in the literature (e.g. by Nualart and Zakai [6], Solé and Utzet [7], Dawidowicz and Twardowska [4]).

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2. Definitions and notation

Let H and H_1 be real separable Hilbert spaces with the norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$ and the scalar products $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{H_1}$, respectively. Let $L_2(H,H_1)$ be a space of Hilbert–Schmidt operators with the norm $\|\cdot\|_{H-S}$.

We consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F})_{t\in[0,T]},P)$ on which an increasing and right-continuous family $(\mathcal{F})_{t\in[0,T]}$ of complete sub-σ-algebras of $\mathcal{F}$ is defined.

We take an H-valued Wiener process $w(t)$, $t\in[0,T]$, with the covariance operator $Q\in L(H)=L(H,H)$. $L(H,H_1)$ denotes a space of bounded linear operators from H to H_1. It is known (Curtain and Pritchard [2], Ch. 5, Da Prato and Zabczyk [3], Ch. 4) that there are real-valued independent Wiener processes $\{w_i(t)\}_{i=0}^{\infty}$ on $[0,T]$ such that

$$w(t)=\sum_{i=0}^{\infty} w_i(t)e_i$$

almost everywhere in $(t,\omega)\in[0,T]\times\Omega$, where $\{e_i\}_{i=0}^{\infty}$ is an orthonormal basis of eignvectors of Q corresponding to eignevalues $\{\lambda_i\}_{i=0}^{\infty}$, $\sum_{i=0}^{\infty}\lambda_i<\infty$. We have

$$E[\Delta w_i\Delta w_j]=(t-s)\lambda_i\delta_{ij} \tag{2.1}$$

for $\Delta w_i=w_i(t)-w_i(s)$ and $s<t$ (δ_{ij} is the Kronecker's delta).

We define (see Chojnowska-Michalik [1], p. 10)

$$\begin{aligned}&\Lambda_T(w,H,H_1)\\&=\Bigg\{\bar{\Psi}\colon[0,T]\times\Omega\to L(H,H_1)\colon\ \bar{\Psi}\text{ is a point-progressively measurable process,}\\&\qquad E\left[\int_0^T\|\bar{\Psi}Q^{\frac12}\|_{H-S}^2\,ds\right]=\|\bar{\Psi}\|_{\Lambda_T}^2=\sum_{i=0}^{\infty}E\left[\int_0^T\|\bar{\Psi}(s,\omega)e_i\|_{H_1}^2\,ds\right]<\infty\Bigg\}.\end{aligned}$$

It the last sum λ_i is omitted because of property (2.1). It is known (Curtain and Pritchard [2], p. 136–143) that for $\bar{\Psi}\in\Lambda_T$ the Itô stochastic integral is well defined and it can be represented by

$$\int_0^T\bar{\Psi}(s,\omega)\,dw(s)=\sum_{i=0}^{\infty}\int_0^T\bar{\Psi}(s,\omega)e_i\,dw_i(s). \tag{2.2}$$

The convergence in (2.2) is in $L^2(\Omega,H_1)$ for each $t>0$.

We proceed to study the stochastic differential equation

$$dz(t)=Az(t)dt+\mathcal{C}(z(t))dt+\mathcal{B}(z(t))dw(t),\qquad z(0)=z_0, \tag{2.3}$$

where:

(A1) $(z(t))_{t\in[0,T]}$ is an H_1-valued stochastic process, $(w(t))_{t\in[0,T]}$ is an H-valued Wiener process with the covariance operator Q, $A: H_1 \supset D(A) \longrightarrow H_1$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$, $\mathcal{C}: H_1 \longrightarrow H_1$ and $\mathcal{B}: H_1 \longrightarrow L(H, H_1)$ are possibly unbounded nonlinear operators. Moreover, we assume that $(S(t))_{t\geq 0}$ is a semigroup of contraction type, i.e., there exists a constant $\beta \in \mathbb{R}_+$ such that $\|S(t)\|_{H_1} \leq \exp(\beta t)$ for all $t \in [0,T]$,

(A2) z_0 is a $D(A)$-valued square integrable $\mathcal{F}_0$-measurable initial random variable,

(A3) there is a real number $K > 0$ such that

$$\|\mathcal{C}(h_1)\|_{H_1}^2 + tr(\mathcal{B}(h_1)Q\mathcal{B}^*(h_1)) \leq K(1 + \|h_1\|_{H_1}^2),$$
$$\|\mathcal{C}(h_1) - \mathcal{C}(\widetilde{h}_1)\|_{H_1}^2 + tr((\mathcal{B}(h_1) - \mathcal{B}(\widetilde{h}_1))Q(\mathcal{B}(h_1) - \mathcal{B}(\widetilde{h}_1))^*) \leq K\|h_1 - \widetilde{h}_1\|_{H_1}^2$$

for $h_1, \widetilde{h}_1 \in H_1$, where "$*$" denotes the adjoint operator,

(A4) the operator $\mathcal{B} \in C^{1,b}$, i.e., is of class C^1 with the bounded derivative. This derivative is assumed to be globally Lipschitzean.

In addition to (2.3) we consider the equation

$$\begin{aligned} d\hat{z}(t) &= A\hat{z}(t)dt + \mathcal{C}(\hat{z}(t))dt + \mathcal{B}(\hat{z}(t))dw(t) \\ &\quad + \frac{1}{2}\widetilde{tr}(QD\mathcal{B}(\hat{z}(t))\mathcal{B}(\hat{z}(t)))dt \\ \hat{z}(0) &= z_0\,, \end{aligned} \tag{2.4}$$

where $\frac{1}{2}(\widetilde{tr}QD\mathcal{B}(\hat{z}(t))\mathcal{B}(\hat{z}(t)))$ is the so-called correction term and is defined below (compare Doss [5], Twardowska [10]).

We observe that the Fréchet derivative $D\mathcal{B}(h_1)$ is in $L(H_1, L(H, H_1))$ for $h_1 \in H_1$. We consider the composition $D\mathcal{B}(h_1) \circ B(h_1) \in L(H, L(H, H_1))$. Let $\Psi \in L(H, L(H, H_1))$ and define

$$\mathcal{B}_{\widetilde{h}_1}(h, h') = \langle \Psi(h)(h'), \widetilde{h}_1 \rangle_{H_1} \in \mathbb{R} \qquad \text{for } h, h' \in H.$$

By the Riesz theorem for the form Ψ on H we conclude that for every $\widetilde{h}_1 \in H_1$ there exists a unique operator $\widetilde{\Psi}(\widetilde{h}_1) \in L(H)$ such that for all $h, h' \in H$,

$$\mathcal{B}_{\widetilde{h}_1}(h, h') = \langle \widetilde{\Psi}(\widetilde{h}_1)(h), h' \rangle_H = \langle \Psi(h)(h'), \widetilde{h}_1 \rangle_{H_1}.$$

But the covariance operator $Q \in L(H)$ has finite trace and therefore the mapping

$$\widetilde{\xi}: \widetilde{h}_1 \in H_1 \longrightarrow tr(Q\widetilde{\Psi}(\widetilde{h}_1)) \in \mathbb{R}$$

is a linear bounded functional on H_1. Therefore, using the Riesz theorem we find a unique $\widetilde{\widetilde{h}}_1 \in H_1$ such that $\widetilde{\xi}(\widetilde{h}_1) = \langle \widetilde{\widetilde{h}}_1, \widetilde{h}_1 \rangle_{H_1}$. Define

$$\widetilde{\widetilde{h}}_1 = \widetilde{tr}(Q\Psi).$$

We observe that $\langle\widetilde{\widetilde{h}}_1, \widetilde{h}\rangle_{H_1}$ is the trace of the operator $Q\widetilde{\Psi}(\widetilde{h}_1) \in L(H)$ but $\widetilde{tr}(Q\Psi)$ is merely a symbol for $\widetilde{\widetilde{h}}_1$.

Since

$$\begin{aligned} tr(Q\widetilde{\Psi}(\widetilde{h}_1)) &= \sum_{i=0}^{\infty}\langle Q\widetilde{\Psi}(\widetilde{h}_1)e_i, e_i\rangle_H = \sum_{i=0}^{\infty}\langle\widetilde{\Psi}(\widetilde{h}_1)e_i, Q^*e_i\rangle_H \\ &= \sum_{i=0}^{\infty}\langle\widetilde{\Psi}(\widetilde{h}_1)e_i, Qe_i\rangle_H = \sum_{i=0}^{\infty}\langle\Psi(e_i)(Qe_i), \widetilde{h}_1\rangle_H \\ &= \sum_{i=0}^{\infty}\langle\Psi(e_i)(\lambda_i e_i), \widetilde{h}_1\rangle_{H_1}, \end{aligned}$$

taking in particular $\Psi = D\mathcal{B}(h_1)\mathcal{B}(h_1)$ we get

$$\widetilde{\widetilde{h}}_1 = \widetilde{tr}(Q\Psi) = \sum_{i=0}^{\infty}\Psi(e_i)(Qe_i) = \sum_{i=0}^{\infty}[D\mathcal{B}(h_1)\mathcal{B}(h_1)(e_i)](\lambda_i e_i). \tag{2.5}$$

We rewrite (2.3) in the mild integral form

$$z(t) = S(t)z_0 + \int_0^t S(t-s)\mathcal{C}(z(s))\,ds + \int_0^t S(t-s)\mathcal{B}(z(s))\,dw(s). \tag{2.6}$$

Similarly, from (2.4) we get

$$\begin{aligned} \hat{z}(t) &= S(t)z_0 + \int_0^t S(t-s)\mathcal{C}(\hat{z}(s))\,ds + \int_0^t S(t-s)\mathcal{B}(\hat{z}(s))\,dw(s) \\ &\quad - \frac{1}{2}\int_0^t S(t-s)\widetilde{tr}(QD\mathcal{B}(\hat{z}(s))\mathcal{B}(\hat{z}(s)))\,ds. \end{aligned} \tag{2.7}$$

First we observe that under our assumptions the integrals are well defined. We have the following definition (see Da Prato and Zabczyk, Ch. 7, § 7.1)

DEFINITION 1. Suppose we are given an H_1-valued initial random variable z_0 and an H-valued Wiener process $(w(t))_{t\in[0,T]}$. Moreover, assume that an H_1-valued stochastic process $(z(t))_{t\in[0,T]}$ has the following properties:

(i) $(z(t))_{t\in[0,T]}$ is progressively measurable,

(ii) $B(z(\cdot)) \in \Lambda_T(w, H, H_1)$,

(iii) for every $t \in [0,T]$ equation (2.6) is satisfied P-almost surely.

Then $(z(t))_{t\in[0,T]}$ is called a mild solution to equation (2.3) with the initial condition z_0.

The uniqueness of solutions is understood in the sense of trajectories.

It is well known that (A1)–(A4) ensure the existence and uniqueness of mild solutions to equations (2.3) and (2.4) (see Twardowska [10]). It only remains to notice that under condition (A4) the term $\widetilde{tr}(QD\mathcal{B}(\hat{z}(t))\mathcal{B}(\hat{z}(t)))$ satisfies condition (A3) because the series in (2.5) converges.

3. The Stratonovich representation of integrals

Let us start from

DEFINITION 2. We define the Stratonovich integral for an operator $\Phi\colon [0,T] \times H_1 \longrightarrow L(H, H_1)$ by

$$
\begin{aligned}
&(\widetilde{S}) \int_0^T \Phi(t, z(t))\, dw(t) \\
&\quad = \lim_{n\to\infty} \widetilde{S}_n \\
&\quad = \lim_{n\to\infty} \sum_{j=1}^{n} \Phi\left(\frac{1}{2}(t_j^n + t_{j-1}^n), \frac{1}{2}(z(t_j^n) + z(t_{j-1}^n))\right)(w(t_j^n) - w(t_{j-1}^n)),
\end{aligned}
\tag{3.1}
$$

where $(w(t))_{t\in[0,T]}$ is an H-valued Wiener process, $(z(t))_{t\in[0,T]}$ is the mild solution to (2.3). The limit is understood P-almost surely and $a = t_0^n < t_1^n < \ldots < t_n^n = b$ is a partition of interval $[a, b]$. We assume that the sequence of partitions is such that $h_n = \max\{t_j^n - t_{j-1}^n,\ j = 1, \ldots, n\} \to 0$ as $n \to \infty$, and the limit does not depend on the choice of the partition. The operator Φ is continuous with respect to the first variable and it has the same properties as the operator $\mathcal{B}$ in § 2 with respect to the second one.

We recal the definition of the Itô integral:

$$
\begin{aligned}
&(\widetilde{\mathcal{I}}) \int_0^T \Phi(t, z(t))\, dw(t) \\
&\quad = \lim_{n\to\infty} \widetilde{\mathcal{I}}_n \\
&\quad = \lim_{n\to\infty} \sum_{j=1}^{n} \Phi(t_{j-1}^n, z(t_{j-1}^n))(w(t_j^n) - w(t_{j-1}^n)),
\end{aligned}
\tag{3.2}
$$

where the same assumptions as in Definition 2 are satisfied, the limit is understood P-almost surely.

Moreover, this integral is a continuous, square integrable H_1-valued martingale (see Da Prato and Zabczyk [3], Ch. 4).

Put $\Delta_j^n w_i = w_i(t_j^n) - w_i(t_{j-1}^n)$. We have

$$
\widetilde{\mathcal{I}}_n = \sum_{i=0}^{\infty} \Phi(t_{j-1}^n, z(t_{j-1}^n)) e_i \Delta_j^n w_i\,,
\tag{3.3}
$$

$$\widetilde{\mathcal{S}}_n = \sum_{i=0}^{\infty}\sum_{j=1}^{n} \Phi\left(\frac{1}{2}(t_j^n + t_{j-1}^n), \frac{1}{2}(z(t_j^n) + z(t_{j-1}^n))\right) e_i \Delta_j^n w_i \,. \tag{3.4}$$

4. The main theorem

The following lemma is valid:

LEMMA 1. *Consider the operator* $\Phi\colon [0,T] \times H_1 \longrightarrow L(H, H_1)$ *satysfying the assumptions of Definition 2 and Lipschitzean with respect to the first variable, uniformly with respect to the second one. Assume that* $\sup_{0\le t\le T} |D_z\Phi(t,z)|$ *is bounded. Let* $(z(t))_{t\in[0,T]}$ *be the mild solution to the stochastic differential equation* (2.3) *with an* H*-valued Wiener process* $(w(t))_{t\in[0,T]}$. *Then the Stratonovich integral* (3.1) *exists and the following relation is satisfied:*

$$\begin{aligned}(\widetilde{\mathcal{S}}) &\int_0^t \Phi(s, z(s))\, dw(s) \\ &= (\widetilde{\mathcal{I}}) \int_0^t \Phi(s, z(s))\, dw(s) + \frac{1}{2}\int_0^t \widetilde{tr}(QD_z\Phi(s, z(s))\mathcal{B}(z(s)))\, ds.\end{aligned} \tag{4.1}$$

PROOF. We examine the difference putting $\frac{1}{2}(t_j^n + t_{j-1}^n) = t_{j-1}^n + \frac{1}{2}(t_j^n - t_{j-1}^n)$

$$\begin{aligned}\widetilde{\mathcal{S}}_n - \widetilde{\mathcal{I}}_n = \sum_{i=0}^{\infty}\sum_{j=1}^{n} &\left[\Phi\left(t_{j-1}^n + \frac{1}{2}(t_j^n - t_{j-1}^n), z(t_{j-1}^n) + \frac{1}{2}(z(t_j^n) - z(t_{j-1}^n))\right)\right.\\ &- \Phi\left(t_{j-1}^n, z(t_{j-1}^n) + \frac{1}{2}(z(t_j^n) - z(t_{j-1}^n))\right)\\ &+ \Phi\left(t_{j-1}^n, z(t_{j-1}^n) + \frac{1}{2}(z(t_j^n) - z(t_{j-1}^n))\right)\\ &\left.- \Phi(t_{j-1}^n, z(t_{j-1}^n))\right]\Delta_j^n w_i e_i\\ = \Phi\left(\frac{1}{2}(t_j^n + t_{j-1}^n), \frac{1}{2}(z(t_j^n) + z(t_{j-1}^n))\right) &- \Phi\left(t_{j-1}^n, \frac{1}{2}(z(t_j^n) + z(t_{j-1}^n))\right)\\ + \sum_{i=0}^{\infty}\sum_{j=1}^{n} &D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))\frac{1}{2}(z(t_j^n) - z(t_{j-1}^n))\Delta_j^n w_i e_i\\ + \sum_{i=0}^{\infty}\sum_{j=1}^{n} &r(t_{j-1}^n, z(t_j^n) - z(t_{j-1}^n))\left\|z(t_j^n) - z(t_{j-1}^n)\right\|_{H_1}\Delta_j^n w_i e_i\\ = I_1 + I_2 + I_3\,,&\end{aligned}$$

where

$$\lim_{z\to 0}\ \sup_{0\le t\le T} |r(t,z)| = 0. \tag{4.2}$$

We recall that the partition is taken now on the interval $[0,T]$ instead of $[a,b]$.

Notice that the series in I_1 converges to zero as $n \to \infty$ (because of the Lipschitz condition for Φ with respect to the first variable) and I_3 converges to zero as $n \to \infty$ with probability one from (4.2).

To compute I_2, first we observe that from (2.6) we have

$$z(t_j^n) - z(t_{j-1}^n) = (S(h^n) - I)z(t_{j-1}^n) + \int_{t_{j-1}^n}^{t_j^n} S(t_{j-1}^n - s)\mathcal{C}(z(s))\,ds + \int_{t_{j-1}^n}^{t_j^n} S(t_{j-1}^n - s)\mathcal{B}(z(s))\,dw(s)$$

so

$$\begin{aligned}
I_2 &= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))(S(h^n) - I)z(t_{j-1}^n)\Delta_j^n w_i e_i \\
&= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n)) \int_{t_{j-1}^n}^{t_j^n} S(t_{j-1}^n - s)\mathcal{C}(z(s))\,ds\,\Delta_j^n w_i e_i \\
&\quad + \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n)) \int_{t_{j-1}^n}^{t_j^n} S(t_{j-1}^n - s)\mathcal{B}(z(s))\,dw(s)\,\Delta_j^n w_i e_i \\
&= \frac{1}{2}(I_{21} + I_{22} + I_{23}).
\end{aligned}$$

We shall show that the terms I_{21} and I_{22} tend to zero and from I_{23} we shall obtain the correction term. Let $c_0, \dots, c_3$ be some positive constants. We have

$$\|I_{21}\|_{H_1} \le c_0 \sum_{j=1}^{n} \|(S(h^n) - I)z(t_{j-1}^n)\|_{H_1} \|\Delta_j^n w\|_H .$$

We recall that

$$\left\|\Delta_j^n w\right\|_H = \left\|\sum_{i=0}^{\infty} (\Delta_j^n w_i)e_i\right\|_H$$

so from the Schwartz inequality, the fact that $E\big[\sum_{j=1}^{n}(\Delta_j^n w_i^2)\big] = t$ and the Chebyshev type inequality we get

$$\begin{aligned}
E\left[\sup_{0\le t\le T} \|I_{21}\|_{H_1}\right] &\le c_0 \sqrt{E\left[\sup_{0\le t\le T} \sum_{j=1}^{n} \|(S(h^n) - I)z(t_{j-1}^n)\|_{H_1}\right]^2} \\
&\quad \times \sqrt{\sum_{j=1}^{n} E\left[\left\|\sum_{i=0}^{\infty}(\Delta_j^n w_i)e_i\right\|_H\right]^2}
\end{aligned}$$

$$\leq c_n \sqrt{\sum_{j=1}^{n} E\left[\left\|\sum_{i=0}^{\infty}(\Delta_j^n w_i)e_i\right\|_H\right]^2}$$
$$= c_n\sqrt{T} \to 0,$$

as $n \to \infty$. Therefore, for each $\varepsilon > 0$

$$P\left(\sup_{0\leq t\leq T}\|I_{21}\|_{H_1} \geq \varepsilon\right) \leq \frac{1}{\varepsilon}E\left[\sup_{0\leq t\leq T}\|I_{21}\|_{H_1}\right] \to 0$$

as $n \to \infty$. Further, we get using (A1) that

$$\|I_{22}\|_{H_1} \leq c_1 \sum_{j=1}^{n} \int_{t_{j-1}^n}^{t_j^n} e^{\beta(t_{j-1}^n - s)}\, ds \cdot \|\Delta_j^n w\|_H$$

and similarly as for I_{21} we obtain for each $\varepsilon > 0$

$$P\left(\sup_{0\leq t\leq T}\|I_{22}\|_{H_1} \geq \varepsilon\right) \leq \frac{1}{\varepsilon}E\left[\sup_{0\leq t\leq T}\|I_{22}\|_{H_1}\right] \to 0$$

as $n \to \infty$.

We transform I_{23} using (1.2) as follows:

$$\begin{aligned}
I_{23}
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n)) \\
&\qquad \times \int_{t_{j-1}^n}^{t_j^n} S(t_{j-1}^n - s)(\mathcal{B}(z(s)) - \mathcal{B}(z(t_{j-1}^n)))e_k\, dw_k(s)\, \Delta_j^n w_i e_i \\
&\quad + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n)) \\
&\qquad \times \left(\int_{t_{j-1}^n}^{t_j^n} (S(t_{j-1}^n - s) - S(h^n))\mathcal{B}(z(t_{j-1}^n))e_k\, dw_k(s)\right)\Delta_j^n w_i e_i \\
&\quad + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))\left(\int_{t_{j-1}^n}^{t_j^n} S(h^n)\mathcal{B}(z(t_{j-1}^n))e_k\, dw_k(s)\right)\Delta_j^n w_i e_i \\
&= I_{231} + I_{232} + I_{233}\,.
\end{aligned}$$

From (A1) and (A3) we get

$$\begin{aligned}
E[\,\|I_{231}\|_{H_1}]
&\leq c_2 \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n} E\left[\sup_{t_{j-1}^n\leq s\leq t_j^n} \|z(s) - z(t_{j-1}^n)\|_{H_1}\|\Delta_j^n w_k e_k\|_H\, \|\Delta_j^n w_i e_i\|_H\right]
\end{aligned}$$

and similarly as for I_{21} we obtain for each $\varepsilon > 0$

$$P\left(\sup_{0\le t\le T}\|I_{231}\|_{H_1}\ge\varepsilon\right)\le\frac{1}{\varepsilon}E\left[\sup_{0\le t\le T}\|I_{231}\|_{H_1}\right]\to 0$$

as $n\to\infty$. Let us notice that

$$\|S(t_{j-1}^n-s)-S(h^n)\|_{H_1}=\|(S(s-t_{j-1}^n)-I)z(t_{j-1}^n)\|_{H_1}\,.$$

From (A3) we get

$$E[\,\|I_{232}\|_{H_1}]$$

$$\le c_3\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n}E\left[\sup_{t_{j-1}^n\le s\le t_j^n}\|(S(s-t_{j-1}^n)-I)z(t_{j-1}^n)\|_{H_1}\|\Delta_j^n w_k e_k\|_H\,\|\Delta_j^n w_i e_i\|_H\right]$$

and similarly as for I_{21} we get for each $\varepsilon > 0$

$$P\left(\sup_{0\le t\le T}\|I_{232}\|_{H_1}\ge\varepsilon\right)\le\frac{1}{\varepsilon}E\left[\sup_{0\le t\le T}\|I_{232}\|_{H_1}\right]\to 0$$

as $n\to\infty$.

Now we transform I_{233} to get the correction term using (2.2) as follows:

$$\begin{aligned}I_{233}&=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n}D_z\Phi(t_{j-1}^n,z(t_{j-1}^n))\left(\int_{t_{j-1}^n}^{t_j^n}S(h^n)\mathcal{B}(z(t_{j-1}^n))e_k\,dw_k(s)\right)\Delta_j^n w_i e_i\\&=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n}D_z\Phi(t_{j-1}^n,z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))\Delta_j^n w_k e_k\Delta_j^n w_i e_i\,.\end{aligned}$$

We estimate the following expression for $i\ne k$ and for $i=k$, separately:

$$\begin{aligned}&\Bigg\|\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{n}D_z\Phi(t_{j-1}^n,z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))\Delta_j^n w_k e_k\Delta_j^n w_i e_i\\&\qquad-\int_0^t\widetilde{tr}(QD_z\Phi(s,z(s))\mathcal{B}(z(s)))\,ds\Bigg\|_{H_1}\\&\le\Bigg\|\sum_{\substack{i,\,k=0\\ i\ne k}}^{\infty}\sum_{j=1}^{n}D_z\Phi(t_{j-1}^n,z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))\Delta_j^n w_k e_k\Delta_j^n w_i e_i\Bigg\|_{H_1}\\&\qquad+\Bigg\|\sum_{i=0}^{\infty}\sum_{j=1}^{n}D_z\Phi(t_{j-1}^n,z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(\Delta_j^n w_i e_i)^2\end{aligned}$$

$$
\left. - \sum_{j=1}^{n} \int\limits_{t_{i-1}^{n}}^{t_i^n} \widetilde{tr}(QD_z\Phi(s,z(s))\mathcal{B}(z(s)))\,ds \right\|_{H_1}
$$

$$
= \|K_1\|_{H_1} + \|K_2\|_{H_1}.
$$

But $K_1 = 0$ because $\{e_i\}_{i=0}^{\infty}$ is an orthonormal basis of eignvectors of Q corresponding to eingevalues $\{\lambda_i\}_{i=0}^{\infty}$ so we only estimate

$$
\begin{aligned}
\|K_2\|_{H_1} \leq & \left\| \sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(\Delta_j^n w_i e_i)^2 \right. \\
& \left. - E\left[\sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(\Delta_j^n w_i e_i)^2\right] \right\|_{H_1} \\
& + \left\| E\left[\sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(\Delta_j^n w_i e_i)^2\right] \right. \\
& \left. - \sum_{j=1}^{n} \int\limits_{t_{i-1}^{n}}^{t_i^n} \widetilde{tr}(QD_z\Phi(s,z(s))\mathcal{B}(z(s)))\,ds \right\|_{H_1} \\
= & \|K_{21}\|_{H_1} + \|K_{22}\|_{H_1}.
\end{aligned}
$$

Now we estimate $E(K_{21})^2$ because $\|K_{21}\|_{H_1} = \sqrt{E(K_{21})^2}$. Using (2.1) and the fact that $\left[\sum_{j=1}^{n} A_j\right]^2 = \sum_{j=1}^{n} A_j^2 + \sum_{j,r=1,j\neq r}^{n} A_j A_r$ we have

$$
\begin{aligned}
& E(K_{21})^2 \\
& = E\left\{ \sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(\Delta_j^n w_i e_i)^2 \right. \\
& \qquad \left. - \sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(t_i^n - t_{i-1}^n)\lambda_i \right\}^2 \\
& = E\left\{ \sum_{j=1}^{n} \left[\sum_{i=0}^{\infty} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))((\Delta_j^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i) \right]^2 \right\} \\
& \quad + E\left\{ \sum_{\substack{j,r=1 \\ j\neq r}}^{n} \left[\sum_{i=0}^{\infty} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))((\Delta_j^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i) \right] \right. \\
& \qquad \left. \times \left[\sum_{i=0}^{\infty} D_z\Phi(t_{r-1}^n, z(t_{r-1}^n))S(h^n)\mathcal{B}(z(t_{r-1}^n))((\Delta_r^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i) \right] \right\} \\
& = K_{211} + K_{212}.
\end{aligned}
$$

Further, using the fact that $E(X) = E(E(X \mid \mathcal{F}))$, we have

$$
\begin{aligned}
K_{211} &= E\left\{ E\left\{ \sum_{j=1}^{n}\sum_{i=0}^{\infty}[D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))]^2 \right.\right. \\
&\qquad \left.\left. \times [(\Delta_j^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i]^2 \right\} \mid \mathcal{F}_{t_{i-1}^n} \right\} \\
&= E\left\{ \left[\sum_{i=0}^{\infty}\sum_{j=1}^{n} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n)) \right]^2 \right. \\
&\qquad \left. \times E[(\Delta_j^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i]^2 \mid \mathcal{F}_{t_{i-1}^n} \right\} \to 0,
\end{aligned} \tag{4.3}
$$

as $n \to \infty$, because

$$
\begin{aligned}
&E[(\Delta_j^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i]^2 \\
&\quad = E(\Delta_j^n w_i)^4 - 2\lambda_i E(\Delta_j^n w_i)^2(t_i^n - t_{i-1}^n) + (t_i^n - t_{i-1}^n)^2{\lambda_i}^2 \\
&\quad = 3(t_i^n - t_{i-1}^n)^2{\lambda_i}^2 - 2(t_i^n - t_{i-1}^n)^2{\lambda_i}^2 + (t_i^n - t_{i-1}^n)^2{\lambda_i}^2 \\
&\quad = 2(t_i^n - t_{i-1}^n)^2{\lambda_i}^2.
\end{aligned}
$$

But

$$
\sum_{j=1}^{n}(t_i^n - t_{i-1}^n)^2 \le \sup_{i=1,\dots,n}(t_i^n - t_{i-1}^n)\sum_{j=1}^{n}(t_i^n - t_{i-1}^n) \to 0,
$$

as $n \to \infty$. Now we write

$$
\begin{aligned}
K_{212} = E\Bigg\{ &\sum_{\substack{j,r=1\\ j\neq r}}^{n}\sum_{i=0}^{\infty}[D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n)) \\
&\times D_z\Phi(t_{r-1}^n, z(t_{r-1}^n))S(h^n)\mathcal{B}(z(t_{r-1}^n))] \\
&\times [((\Delta_j^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i)((\Delta_r^n w_i)^2 - (t_i^n - t_{i-1}^n)\lambda_i)] \Bigg\}
\end{aligned}
$$

and we estimate it similarly as in (4.3). Thus $\sqrt{E(K_{21})^2} \to 0$, as $n \to \infty$.

Now we estimate using (2.1)

$$
\begin{aligned}
\|K_{22}\|_{H_1} \le \Bigg\| &\sum_{j=1}^{n} E\left[\sum_{i=0}^{\infty} D_z\Phi(t_{j-1}^n, z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(\Delta_j^n w_i)^2\right] \\
&- \sum_{j=1}^{n}\int_{t_{i-1}^n}^{t_i^n} \widetilde{tr}(QD_z\Phi(t_{j-1}^n, z(t_{j-1}^n))\mathcal{B}(z(t_{j-1}^n)))\,ds \Bigg\|_{H_1}
\end{aligned}
$$

$$+\left\|\sum_{j=1}^{n}\int_{t_{i-1}^n}^{t_i^n}\widetilde{tr}(QD_z\Phi(t_{j-1}^n,z(t_{j-1}^n))\mathcal{B}(z(t_{j-1}^n)))\,ds\right.$$

$$\left.-\sum_{j=1}^{n}\int_{t_{i-1}^n}^{t_i^n}\widetilde{tr}(QD_z\Phi(s,z(s))\mathcal{B}(z(s)))\,ds\right\|_{H_1}$$

$$=\|K_{221}\|_{H_1}+\|K_{222}\|_{H_1}\,.$$

Further, from the form of the correction term we have

$$\|K_{221}\|_{H_1}=\left\|E\left[\sum_{i=0}^{\infty}\sum_{j=1}^{n}D_z\Phi(t_{j-1}^n,z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(\Delta_j^n w_i e_i)^2\right]\right.$$

$$\left.-\sum_{j=1}^{n}\int_{t_{i-1}^n}^{t_i^n}\widetilde{tr}(QD_z\Phi(t_{j-1}^n,z(t_{j-1}^n))\mathcal{B}(z(t_{j-1}^n)))\,ds\right\|_{H_1}$$

$$=\left\|\sum_{j=1}^{n}\left[\sum_{i=0}^{\infty}D_z\Phi(t_{j-1}^n,z(t_{j-1}^n))S(h^n)\mathcal{B}(z(t_{j-1}^n))(t_j^n-t_{j-1}^n)\lambda_i\right.\right.$$

$$\left.\left.-\widetilde{tr}(QD_z\Phi(t_{j-1}^n,z(t_{j-1}^n))\mathcal{B}(z(t_{j-1}^n)))(t_j^n-t_{j-1}^n)\right]\right\|_{H_1}\to 0$$

as $n\to\infty$, and finaly $\|K_{22}\|\to 0$, as $n\to\infty$. Thus, using the Chebyshev type inequality we get that for each $\varepsilon>0$

$$P\left(\sup_{0\le t\le T}\|I_{233}\|_{H_1}\ge\varepsilon\right)\le\frac{1}{\varepsilon}E\left[\sup_{0\le t\le T}\|I_{233}\|_{H_1}\right]\to 0,$$

as $n\to\infty$, and therefore we have

$$P\left(\sup_{0\le t\le T}\left\|\widetilde{S}_n-\widetilde{\mathcal{I}}_n-\frac{1}{2}\int_0^t\widetilde{tr}(QD_z\Phi(s,z(s))\mathcal{B}(z(s)))\,ds\right\|_{H_1}\ge\varepsilon\right)$$

$$\le\frac{1}{\varepsilon}E\left[\sup_{0\le t\le T}\left\|\widetilde{S}_n-\widetilde{\mathcal{I}}_n-\frac{1}{2}\int_0^t\widetilde{tr}(QD_z\Phi(s,z(s))\mathcal{B}(z(s)))\,ds\right\|_{H_1}\right]\to 0$$

as $n\to\infty$.

From this we obtain the existence of $\lim_{n\to\infty}\widetilde{S}_n$ and formula (4.1), which completes the proof of Lemma 1. □

REMARK 1. If we put $\Phi(t,z(t))=\mathcal{B}(z(t))$ then the correction term in (4.1) has the form

$$\frac{1}{2}\int_0^t\widetilde{tr}(QD\mathcal{B}(z(s))\mathcal{B}(z(s)))\,ds.$$

It is the same correction term that occurs in the approximation theorem of Wong–Zakai type, which was expected during our considerations.

Let

$$\widetilde{\mathcal{S}}(t) = (\widetilde{S}) \int_0^t \Phi(s, z(s))\, dw(s),$$

$$\widetilde{\mathcal{I}}(t) = (\widetilde{\mathcal{I}}) \int_0^t \Phi(s, z(s))\, dw(s),$$

$$\widetilde{\mathcal{P}}(t) = (\widetilde{\mathcal{P}}) \int_0^t \int_0^t \widetilde{tr}(QD_z\Phi(s, z(s))\mathcal{B}(z(s)))\, ds.$$

Now we can prove the following

THEOREM 1. *Consider the operator* $\Phi: [0,T] \times H_1 \longrightarrow L(H, H_1)$ *satysfying the assumptions of Definition 2 and Lipschitzean with respect to the first variable, uniformly with respect to the second one. Assume that* $\sup_{0\leq t\leq T} |D_z\Phi(t,z)|$ *is bounded. Let* $(z(t))_{t\in[0,T]}$ *be the mild solution to the stochastic differential equation* (2.3) *with an* H*-valued Wiener process* $(w(t))_{t\in[0,T]}$. *Then, the following relation is satisfied*

$$\begin{aligned}
&(\widetilde{S}) \int_0^t S(t-s)\Phi(s, z(s))\, dw(s)\\
&\quad = (\widetilde{\mathcal{I}}) \int_0^t S(t-s)\Phi(s, z(s))\, dw(s)\\
&\qquad + \frac{1}{2} \int_0^t S(t-s)\widetilde{tr}(QD_z\Phi(s, z(s))\mathcal{B}(z(s)))\, ds.
\end{aligned} \tag{4.4}$$

PROOF. We have from Lemma 1

$$\widetilde{\mathcal{S}}(t) = \widetilde{\mathcal{I}}(t) + \widetilde{\mathcal{P}}(t).$$

Now we define some integrals from deterministic operator functions for arbitrary intermediate points $\tilde{s}_j$ as follows:

$$\begin{aligned}
&(\widetilde{S}) \int_0^t S(t-s)\Phi(s, z(s))\, dw(s)\\
&= \widetilde{I}_1\\
&= \lim_{n\to\infty} \sum_{j=1}^n S(t-\tilde{s}_j)[\widetilde{\mathcal{S}}(s_j) - \widetilde{\mathcal{S}}(s_{j-1})],
\end{aligned}$$

$$
\begin{aligned}
(\widetilde{\mathcal{I}}) \int_0^t S(t-s)\Phi(s,z(s))\,dw(s)
&= \widetilde{I}_2 \\
&= \lim_{n\to\infty}\sum_{j=1}^{n} S(t-\tilde{s}_j)[\widetilde{\mathcal{I}}(s_j) - \widetilde{\mathcal{I}}(s_{j-1})],
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\int_0^t S(t-s)\widetilde{tr}(QD_z\Phi(s,z(s))\mathcal{B}(z(s)))\,ds
&= \widetilde{I}_3 \\
&= \lim_{n\to\infty}\sum_{j=1}^{n} S(t-\tilde{s}_j)[\widetilde{\mathcal{P}}(s_j) - \widetilde{\mathcal{P}}(s_{j-1})].
\end{aligned}
$$

We also have from Lemma 1

$$
\widetilde{\mathcal{S}}(s_j) - \widetilde{\mathcal{S}}(s_{j-1}) = \widetilde{\mathcal{I}}(s_j) - \widetilde{\mathcal{I}}(s_{j-1}) + \widetilde{\mathcal{P}}(s_j) - \widetilde{\mathcal{P}}(s_{j-1})
$$

and then

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{j=1}^{n} S(t-\tilde{s}_j)[\widetilde{\mathcal{S}}(s_j) - \widetilde{\mathcal{S}}(s_{j-1})] &= \lim_{n\to\infty}\sum_{j=1}^{n} S(t-\tilde{s}_j)[\widetilde{\mathcal{I}}(s_j) - \widetilde{\mathcal{I}}(s_{j-1})] \\
&\quad + \lim_{n\to\infty}\sum_{j=1}^{n} S(t-\tilde{s}_j)[\widetilde{\mathcal{P}}(s_j) - \widetilde{\mathcal{P}}(s_{j-1})].
\end{aligned}
$$

Therefore, we obtain (4.4), which completes the proof. □

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