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ON GENERALIZED SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS

Abstract. We prove theorem on the existence and uniqueness of the distributional solutions of the Cauchy problem for equation (1.0).

1. Introduction. In this note we consider the following equation

$$y' = F(y, y(h)), \tag{1.0}$$

where F is a given operation, y is an unknown real function of locally bounded variation in $\mathbf{R}^1$ ($\mathbf{R}^1$ denotes the real line), h is a continuous real function defined in $\mathbf{R}^1$ and $F(y, y(h))$ is a measure. The derivative is understood in the distributional sense. Our theorems generalize some results given in [2], [3] and [4].

2. Notation. By $\mathscr{V}$ ($\mathscr{V}[t_0, t+a)$) we denote the set of all real functions of locally bounded variation in $\mathbf{R}^1$ (resp. the set of real functions of locally bounded variation defined in the interval $[t_0, t_0+a)$). We say that a distribution p is a *measure in* $\mathbf{R}^1$ if p is the first distributional derivative of a function from the class $\mathscr{V}$. The symbol $\mathscr{M}$ ($\tilde{\mathscr{M}}$) denotes the set of all measures (resp. non negative measures) defined in $\mathbf{R}^1$. Let $P \in \mathscr{V}$. Then we define

$$P^*(t_0) = \frac{P(t_0+)+P(t_0-)}{2}, \tag{2.0}$$

$$\int_c^d p(t)\,dt = P^*(d)-P^*(c) \tag{2.1}$$

and

$$\int_{-\infty}^{\infty} p(t)\,dt = \lim_{c\to-\infty}\left(\lim_{d\to\infty}\int_c^d p(t)\,dt\right), \tag{2.2}$$

where $P(t_0+)$, $(P(t_0-))$ denotes the right (resp. left) hand side limits of the function P at the point t_0 and $P' = p$. One may show that if $Q \in \mathscr{V}$ and $p \in \mathscr{M}$, then $p\cdot Q \in \mathscr{M}$ (see [1]) and

$$|pQ| \leqslant |p|\,|Q|, \tag{2.3}$$

$$\left|\int_c^d p(t)\,Q(t)\,dt\right| \leqslant \sup_{c\leqslant t\leqslant d} |Q|^*(t)\int_c^d |p|(t)\,dt, \tag{2.4}$$

$$\int_c^d p(t)\,dt \leqslant \int_c^d q(t)\,dt, \tag{2.5}$$

where $q \in \mathscr{M}$ and $p \leqslant q$ (see [5], [6]). By $\mathscr{V}^*$ we denote the set of all functions $z \in \mathscr{V}$ such that $z(t) = z^*(t)$ for every t. Let $L \in \tilde{\mathscr{M}}$ and c be a positive constant. We define

$$\mathscr{B}_L^c = \{x \in \mathscr{V}^*: \sup_{-\infty<t<\infty}\left[\left(|x|^*(t_0)+\mathrm{var}_{t_0}^t x(s)\right)E(t)\right] < \infty\}, \tag{2.6}$$

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where $\text{var}_{t_0}^t x = \text{var}_{t_0}^t x$ if $t < t_0$, $\text{var}_0^0 x = 0$ and $E(t) = e^{-c|\int_{t_0}^t L(s)ds|}$. The set $\mathscr{B}_L^c$ is a linear space (the sum of two functions and the product of a scalar and a function is understood in the usual way). Next, if $x \in \mathscr{B}_L^c$ we put

$$w(t) = |x|^*(t_0) + \text{var}_{t_0}^t x(s), \tag{2.7}$$

$$E^{-1}(t) = \big(E(t)\big)^{-1}, \tag{2.8}$$

$$\|x\| = \sup_{-\infty < t < \infty} w(t)E(t), \tag{2.9}$$

$$\|x\|_{[a,b]} = \sup_{a \leqslant t \leqslant b} w(t)E(t), \qquad (t_0 \in [a,b]) \tag{2.10}$$

$$\|x\|^* = \sup_{-\infty < t < \infty} w(t) \tag{2.11}$$

and

$$\|x\|_{[a,b]}^* = \sup_{a \leqslant t \leqslant b} w(t), \qquad t_0 \in [a,b]. \tag{2.12}$$

One may show that a $\|\,.\,\|$ is a norm in $\mathscr{B}_L^c$. The space $\mathscr{B}_L^c$ with the norm (2.9) we denote by $\mathscr{B}$.

3. The main results.

THEOREM 3.1. *The space $\mathscr{B}$ is a Banach space.*

Now we examine the following problem

$$\begin{cases} y' = F\big(y, y(h)\big) & (3.0) \\ y^*(t_0) = \bar{y}_0. & (3.1) \end{cases}$$

By a solution of the problem (3.0)—(3.1) we understand a function $y \in \mathscr{B}$ which satisfies (3.0) (in the distributional sense) and (3.1). We shall introduce two hypotheses.

Hypothesis H_1.

1. F is an operation defined for every system of functions (u, v) of the class $\mathscr{V}$.
2. $F(u, v) \in \mathscr{M}$.
3. h is the continuous real function defined in $\mathbf{R}^1$ such that if $u \in \mathscr{V}$, then $u(h) \in \mathscr{V}$.
4. For every M_0 there exists N such that $0 < N < M_0$ and

$$\Big\|\int_{t_0}^t \big|F\big(y, y(h)\big)\big|(s)ds\Big\|^* \leqslant N \text{ for } t \in (-\infty, \infty),$$

whenever $\|y\|^* \leqslant M_0$.

5. $|\bar{y}_0| \leqslant M_0 - N$.
6. If $y_n, y_0 \in \mathscr{B}$, $\|y_n\|^* \leqslant M_0$ $(n = 0.1, 2, \ldots)$ and $y_n \rightrightarrows y_0$ (almost uniformly), then

$$\lim_{n\to\infty} \|T(y_n) - T(y_0)\| = 0,$$

where

$$T(y_i)(t) = \bar{y}_0 + \int_{t_0}^{t} F(y_i, y_i(h))(s)\mathrm{d}s \qquad (i = 1, 2, \ldots),$$

7. There exists $k \in \tilde{\mathscr{M}}$ such

$$|F(y, y(h))| \leqslant k$$

for every $y \in \mathscr{V}$ such that $\|y\|^* \leqslant M_0$ and $\|\hat{k}\|^* \leqslant M_0$, where $(\hat{k})' = k$.

EXAMPLE 1. Let $\lim\limits_{t \to \infty} (\hat{L})^*(t) = \infty$ and $\lim\limits_{t \to \infty} (\hat{L})^*(t) = -\infty$, where $(\hat{L})' = L \in \tilde{M}$. Moreover, let $Ł \in \tilde{\mathscr{M}}$, $\int_{-\infty}^{\infty} Ł(t)\mathrm{d}t = r < \infty$, $|\int_{t_0}^{t} Ł(s)\mathrm{d}s|^*(t_0) = m$, $0 < r+m < 1$, $\bar{h}$ a constant and $y \in \mathscr{V}$. It is not difficult to check that the operations F_1 and F_2 defined as follows

$$F_1(y, y(h))(t) := Ł(t)y(t+\bar{h}),$$

$$F_2(y, y(h))(t) := Ł(t)\frac{y(t)}{1+|y(t+\bar{h})|}$$

satisfy the hypothesis H_1. In fact, by (2.4) we can write

$$\|F_j(y, y(h))\|^* \leqslant M_0(m+r) := N < M_0$$

for $j = 1, 2,$ $0 < m+r < 1$ and $\|y\|^* \leqslant M_0$. Let $y_n, y_0 \in \mathscr{B}$, $\|y_n\|^* \leqslant M_0$ $(n = 0, 1, 2, \ldots)$ and let $y_n \rightrightarrows y_0$. Then we have

$$\|T(y_i) - T(y_0)\| = \|\int_{t_0}^{t} [F_j(y_i, y_i(h)) - F_j(y_0, y_0(h))](s)\mathrm{d}s\| \leqslant$$

$$\leqslant \|\int_{t_0}^{t} [F_j(y_i, y_i(h)) - F_j(y_0, y_0(h))](s)\mathrm{d}s\|_{[-a,a]} + \frac{4M_0 r(1+M_0 D_j)}{\mathrm{e}^{cP(a)}}$$

where $j = 1, 2,$ $P(a) = \min[|\hat{L}^*(a) - \hat{L}^*(t_0)|, \ |\hat{L}^*(-a) - \hat{L}^*(t_0)|]$, $t_0 \in [-a, a]$, $D_1 = 0$ and $D_2 = 1$. Thus

$$\|T(y_n) - T(y_0)\| < \varepsilon \text{ for } i > n_0 \text{ and } \varepsilon > 0$$

(and for sufficiently large a). We put

$$k(t) := M_0 Ł(t).$$

EXAMPLE 2. Let $f: \mathbf{R}^3 \to \mathbf{R}^1$ be a continuous function such that

$$|f(t, y(t), y(h(t)))| \leqslant Q(t)$$

whenever $\|y\|^* \leqslant M_0$ and $\int_{-\infty}^{\infty} Q(t)\mathrm{d}t \leqslant M_0$. Then we consider operation F defined as follows

$$F(y, y(h))(t) := f(t, y(t), y(h(t))).$$

Next, we assume that

1. For every M_0 there exists N such that $0 < N < M_0$ and

$$\Big\|\int_{t_0}^{t} |F(y, y(h))|(s)\mathrm{d}s\Big\|^* \leqslant N,$$

whenever $\|y\|^* \leqslant M_0$.

2. h is a continuous real non increasing function.
3. $|\bar{y}_0| \leqslant M_0 - N$.

It is not difficult to verify that the operation F satisfies assumptions of hypothesis H_1.

Hypothesis H_2.

1. Assumptions 1. and 2. of H_1 are fulfilled.
2. h is a real continuous function such that for every $u, v \in \mathscr{V}$ and t holds

$$|u(h) - v(h)|^*(t) \leqslant |u - v|^*(t_0) + \mathrm{var}_{t_0}^{t+\gamma(t)}(u - v)^*(s)$$

and $u(h) \in \mathscr{V}$, where γ is a continuous real function defined in $(-\infty, \infty)$.

3. There exists $L \in \mathscr{M}$ such that for every $u, v, \bar{u}, \bar{v} \in \mathscr{V}$ holds

$$|F(u, v) - F(\bar{u}, \bar{v})| \leqslant L(|u - \bar{u}| + |v - \bar{v}|),$$

where

$$\int_{-\infty}^{\infty} L(t)\mathrm{d}t = r, \qquad |F(0,0)| \leqslant cL, \qquad c > 0 \quad \text{and} \quad \Big|\int_{t_0}^{t} L(s)\mathrm{d}s\Big|^*(t_0) = q.$$

4. $\sup\limits_{-\infty < t < \infty} \mathrm{e}^{c\left|\int_t^{t+\gamma(t)} L(s)\mathrm{d}s\right|} = m.$

5. $\alpha := (q + r)(m + 1) < 1$.

6. $p \geqslant \dfrac{|\bar{y}_0| + cq + 1}{1 - (q + r)(m + 1)}$.

7. $\mathscr{B}^* := \{y \in \mathscr{B}: \|y\|^* \leqslant p\}$.

EXAMPLE 3. We consider the following problem

$$y' = \frac{1}{4}\delta(t)y(t + \bar{h}), \qquad y^*(0) = 1, \tag{3.2}$$

where δ denotes the Dirac delta, $\bar{h}$ a constant. If we shall put

$$L = \frac{1}{4}\delta, \qquad \gamma(t) = \bar{h}, \qquad r = \frac{1}{4}, \qquad q = \frac{1}{8}, \qquad m = \mathrm{e}^{\frac{1}{4}}, \qquad \alpha < 1$$

and

$$F(y, y(h))(t) = \frac{1}{4}\delta(t)y(t + \bar{h}),$$

then hypothesis H_2 is satisfied.

THEOREM 3.2. *Let hypothesis* H_1 *be fulfilled. Then the problem* (3.0)—3.1) *has at least one solution.*

THEOREM 3.3. *Let hypothesis* H_2 *be satisfied. Then the problem* (3.0)—(3.1) *has exactly one solution in the clase* $\mathscr{B}^*$.

4. Proofs.

Proof of Theorem 3.1. Let $y_n \in \mathscr{B}$ $(n = 1, 2, \ldots)$ and let for every $\varepsilon > 0$ there exists r_0 such that

$$\|y_n - y_m\| < \varepsilon \tag{4.0}$$

for every $n, m > r_0$. Then

$$\begin{aligned}(4.1)\quad (|y_n(t) - y_m(t)|)E(t) &= \\ &= (|y_n(t) - y_m(t) + y_n(t_0) - y_n(t_0) + y_m(t_0) - y_m(t_0)|)E(t) \leqslant \\ &\leqslant (|y_n - y_m|^*(t_0) + \mathrm{var}_{t_0}^t(y_n - y_m)(s))E(t) \leqslant \\ &\leqslant \|y_n - y_m\| < \varepsilon \quad (n, m > r_0).\end{aligned}$$

Thus the sequence $\{y_n(t)\}$ is almost uniformly convergent to a function y. We shall show that $y \in \mathscr{B}$. In fact, from (4.1) we have

$$\sup_{-\infty < t < \infty} (\mathrm{var}_{t_0}^t(y_n - y_m)(s))E(t) \leqslant \|y_n - y_m\| < \frac{\varepsilon}{2} \tag{4.2}$$

for $n, m > r_1$. Hence taking into account [7, Theorem 5.7] we infer that

$$\sup_{-\infty < t < \infty} \mathrm{var}_{t_0}^t(y_n - y)(s)E(t) \leqslant \frac{\varepsilon}{2} \text{ for } n > r_1. \tag{4.3}$$

Let

$$|y_n - y|^*(t_0) \leqslant \frac{\varepsilon}{2} \text{ for } n > r_2. \tag{4.4}$$

Then, by (4.3) and (4.4) we can write

$$\|y_n - y\| \leqslant \varepsilon \text{ for } n > r_3, \tag{4.5}$$

where $r_3 = \max(r_1, r_2)$. Thus the proof of Theorem 3.1 is complete.

REMARK. Let $\mathscr{V}^*(a,b)$ be the set of all real functions z of locally bounded variation in the interval $(a, b) \subset \mathbf{R}^1$ such that $z(t) = z^*(t)$ for every $t \in (a, b)$. Moreover, let $L = 0$, $t_0 \in (a, b)$ and let

$$\|x\|_{(a,b)} := |x|^*(t_0) + \sup_{a<t<b} (\mathrm{var}_{t_0}^t x^*(s)).$$

We define

$$\bar{\mathscr{V}}(a, b) := \{x \in \mathscr{V}^*(a, b) \colon \|x\|_{(a,b)} < \infty\}.$$

We conclude that the linear space $\bar{\mathscr{V}}(a, b)$ with the norm $\|x\|_{(a,b)}$ is a Banach space.

Before giving the proof of Theorem 3.2 we shall formulate the properties $\tilde{L}$, L^* and two lemmas.

Let $\mathscr{A} \subset \mathscr{V}[t_0, t_0+a)$ $(0 < a \leqslant \infty)$. We say that a family $\mathscr{A}$ has the property $\tilde{L}$, if the following condition holds (see [8] p. 29)

$$\bigwedge_{\varepsilon>0} \bigwedge_{t_1 \in [t_0, t_0+a)} \bigvee_{\delta>0} \bigwedge_{t \in [t_0, t_0+a)} \bigwedge_{f \in \mathscr{A}} \left[(0 < t-t_1 < \delta \Rightarrow |f(t)-f(t_1+)| < \varepsilon) \wedge (0 < t_1-t < \delta \Rightarrow |f(t)-f(t_1-)| < \varepsilon)\right].$$

LEMMA 4.1. (see [8] p. 30). *Let $f_n \in \mathscr{V}[t_0, t_0+a)$, $n = 0, 1, 2, \ldots$. If the sequence $\{f_n\}$ has the property $\tilde{L}$ and if $f_n \to f_0$ for every t, then $f_n \to f_0$ almost uniformly.*

We assume that $\mathscr{A} \subset \mathscr{B}$. We say that a family $\mathscr{A}$ has the property L^* if the following condition holds

$$\bigwedge_{\varepsilon>0} \bigwedge_{t_1 \in (-\infty, \infty)} \bigvee_{\delta>0} \bigwedge_{t \in (-\infty, \infty)} \bigwedge_{f \in \mathscr{A}} \left[(0 < t-t_1 < \delta \Rightarrow ||f(t)-f(t_1+)| < \varepsilon) \wedge (0 < t_1-t < \delta \Rightarrow |f(t)-f(t_1-)| < \varepsilon)\right].$$

From Lemma 4.1. we conclude

LEMMA 4.2. *Let $f_n \in \mathscr{B}$, $n = 0, 1, 2, \ldots$. If the sequence $\{f_n\}$ has the property L^* and if $f_n \to f_0$ for every t, then $f_n \to f_0$ almost uniformly in $(-\infty, \infty)$.*

Proof of Theorem 3.2. We shall apply Schauder's — Mazur's theorem on fixed point. In this purpose we consider the set $\mathscr{U}^* \subset \mathscr{B}$ defined as follows

$$\mathscr{U}^* = \{x \in \mathscr{B} : \|x\|^* \leqslant M_0\}. \tag{4.6}$$

Let $\mathscr{U}$ be the set of all functions $y \in \mathscr{U}^*$ such that

$$|y(t)-y(t_1+)| \leqslant |\hat{k}^*(t)-\hat{k}^*(t_1+)| \text{ for } t > t_1$$

and

$$|y(t)-y(t_1-)| \leqslant |\hat{k}^*(t)-\hat{k}^*(t_1-)| \text{ for } t < t_1.$$

It is easy to observe that $\mathscr{U}$ is non empty set. Let $u, v \in \mathscr{U}$, $0 \leqslant \lambda \leqslant 1$ and $y = \lambda u + (1-\lambda)v$. Then

$$\|y\|^* \leqslant \|\lambda u\|^* + \|(1-\lambda)v\|^* \leqslant M_0$$

and

$$\begin{aligned} |y(t)-y(t_1+)| &\leqslant \lambda|u(t)-u(t_1+)| + (1-\lambda)|v(t)-v(t_1+)| \leqslant \\ &\leqslant \lambda|\hat{k}^*(t)-\hat{k}^*(t_1+)| + (1-\lambda)|\hat{k}^*(t)-\hat{k}^*(t_1+)| \leqslant \\ &\leqslant |\hat{k}^*(t)-\hat{k}^*(t_1+)| \end{aligned}$$

for $t > t_1$. Similarly

$$|y(t)-y(t_1-)| \leqslant |\hat{k}^*(t)-\hat{k}^*(t_1-)| \text{ for } t < t_1.$$

Hence, we infer that $\mathscr{U}$ is a convex set. Moreover, we shall show that $\mathscr{U}$ is a closed set. In fact, let $x_n \in \mathscr{U}$ $(n = 1, 2, \ldots)$ and let $\lim_{n \to \infty} x_n = x$. Then for every $\varepsilon > 0$ there exists a number r_0 such that

(4.7) $$\|x_n - x\|_{[-a,a]} \leqslant \|x_n - x\| < \varepsilon \text{ for } n > r_0.$$

Hence

(4.8) $$\|x_n - x\|^*_{[-a,a]} < \varepsilon \text{ for } n > r_1.$$

From the last inequality, we get

(4.9) $$\|x\|^*_{[-a,a]} < \|x_n\|^*_{[-a,a]} + \varepsilon \leqslant \|x_n\|^* + \varepsilon \leqslant M_0 + \varepsilon$$

and

$$\|x\|^*_{[-a,a]} \leqslant M_0.$$

Hence we can write

(4.10) $$\|x\|^* \leqslant M_0.$$

From the definition of the set $\mathscr{U}$, we have

(4.11) $$|x_n(t) - x_n(t_1+)| \leqslant |\hat{k}^*(t) - \hat{k}^*(t_1+)| \text{ for } t > t_1$$

and

(4.12) $$|x_n(t) - x_n(t_1-)| \leqslant |\hat{k}^*(t) - \hat{k}^*(t_1-)| \text{ for } t < t_1.$$

Since the sequence $\{x_n\}$ is almost uniformly convergent to x, by (4.11) and (4.12) we obtain

(4.13) $$|x^*(t) - x^*(t_1+)| \leqslant |\hat{k}^*(t) - \hat{k}^*(t_1+)| \text{ for } t > t_1$$

and

$$|x^*(t) - x^*(t_1-)| \leqslant |\hat{k}^*(t) - \hat{k}^*(t_1-)| \text{ for } t < t_1.$$

Taking into account relations (4.10) and (4.13) we inter that $\mathscr{U}$ is a closed set. Next, we define transformation T as follows

(4.14) $$T(x)(t) = \bar{y}_0 + \int_{t_0}^{t} F(x, x(h))(s)\,ds := y,$$

where $x \in \mathscr{U}$. Using (4.14) and assumptions 4,5 of H_1 we have

(4.15) $$\|T(x)\|^* \leqslant |\bar{y}_0| + N \leqslant M_0 - N + N \leqslant M_0.$$

Moreover, by 7. of H_1 and (4.14) we can write

$$|y^*(t)-y^*(t_1+)| \leqslant |\hat{k}^*(t)-\hat{k}^*(t_1+)| \tag{4.16}$$

and

$$|y^*(t)-y^*(t_1-)| \leqslant |\hat{k}^*(t)-\hat{k}^*(t_1-)|. \tag{4.17}$$

Applying (4.15), (4.16) and (4.17) we obtain

$$T(\mathscr{U}) \subset \mathscr{U}. \tag{4.18}$$

Let $x_n \in \mathscr{U}$ $(n = 1, 2, \ldots)$ and let $\lim\limits_{n\to\infty} x_n = x$. Taking into account 6. of H_1 and almost uniformly convergence of the sequence $\{x_n\}$, we conclude that T is a continuous operation. In the sequel we shall prove that $T(\mathscr{U})$ is a compact set in $\mathscr{B}$. In fact, let $y_i \in T(\mathscr{U})$ $(i = 1, \ldots)$ i.e.

$$y_i = T(x_i), \qquad x_i \in \mathscr{U}, (i = 1, 2, \ldots). \tag{4.19}$$

The sequence $\{x_i\}$ has the property L^* and

$$\|x_i\|^* \leqslant M_0. \tag{4.20}$$

Applying Helly's theorem and Lemma 4.2 we infer that there exists a subsequence $\{x_{i_q}\}$ of the sequence $\{x_i\}$ almost uniformly convergent to a function $x \in \mathscr{B}$, because (by (4.20) and [7] p. 371)

$$\|x\|^* \leqslant M_0. \tag{4.21}$$

On the other hand from 6. of H_1 we get

$$\lim_{q\to\infty} T(x_{i_q}) = \lim_{q\to\infty} y_{i_q} = T(x) \in \mathscr{B}. \tag{4.22}$$

Thus $T(\mathscr{U})$ is a compact set. Now, we use Schauder's — Mazur's theorem on fixed point to transformation T, which implies our assertion.

P r o o f of Theorem 3.3. We shall apply the Banach theorem on fixed point. In this purpose we consider the operation T defined by (4.14) for $x \in \mathscr{B}$. Next, we consider the set $\mathscr{B}^*$ (defined by 10. of H_2). We shall show that under assumptions H_2

$$T(\mathscr{B}^*) \subset \mathscr{B}^*. \tag{4.23}$$

In fact, let $x \in \mathscr{B}^*$. Then

$$(4.24)\qquad |y|^*(t_0)+\mathrm{var}_{t_0}^t y(s) \leqslant |\bar{y}_0| + \Big|\int_{t_0}^{t} F(x,x(h))(s)\mathrm{d}s\Big|^*(t_0)+$$
$$+\mathrm{var}_{t_0}^t\Big(\int_{t_0}^{s} F(x,x(h))(\tau)\mathrm{d}\tau\Big) \leqslant$$
$$\leqslant |\bar{y}_0| + \Big|\int_{t_0}^{t}\big|F(x,x(h))-F(0,0)\big|(s)\mathrm{d}s\Big|^*(t_0)+$$
$$+\Big|\int_{t_0}^{t}\big|F(0,0)(s)\mathrm{d}s\Big|^*(t_0)+\mathrm{var}_{t_0}^t\Big(\int_{t_0}^{s}\big|F(x,x(h))-$$
$$-F(0,0)\big|(\tau)\mathrm{d}\tau\Big)+\mathrm{var}_{t_0}^t\Big(\int_{t_0}^{s}\big|F(0,0)\big|(\tau)\mathrm{d}\tau \leqslant$$
$$\leqslant |\bar{y}_0| + \Big|\int_{t_0}^{t}(L|x|)(s)\mathrm{d}s\Big|^*(t_0)+\Big|\int_{t_0}^{t}(L|x(h)|)(s)\mathrm{d}s\Big|^*(t_0)+$$
$$+c\Big|\int_{t_0}^{t}L(s)\mathrm{d}s\Big|^*(t_0)+\Big|\int_{t_0}^{t}(L|x|)(s)\mathrm{d}s\Big|+$$
$$+\Big|\int_{t_0}^{t}(L|x(h)|)(s)\mathrm{d}s\Big|+c\Big|\int_{t_0}^{t}L(s)\mathrm{d}s\Big|.$$

Taking into account 2. of H_2 we have

$$(4.25)\qquad |y|^*(0)+\mathrm{var}_{t_0}^t y(s) \leqslant |\bar{y}_0| + \|x\|\,\Big|\int_{t_0}^{t} L(s)E^{-1}(s)\mathrm{d}s\Big|^*(t_0)+$$
$$+\|x\|\int_{t_0}^{t} L(s)e^{c\left|\int_{t_0}^{s+\gamma(s)} L(u)\mathrm{d}u\right|}\mathrm{d}sI^*(t_0)+cq+$$
$$+\|x\|\,\Big|\int_{t_0}^{t} L(s)E^{-1}(s)\mathrm{d}s\,+$$
$$+\|x\|\,\Big|\int_{t_0}^{t} L(s)e^{c\left|\int_{t_0}^{s+\gamma(s)} L(u)\mathrm{d}u\right|}\mathrm{d}s\Big|+E^{-1}(t)\leqslant$$
$$\leqslant |\bar{y}_0|+E^{-1}(t)q\,\|x\|+$$
$$+\|x\|E^{-1}(t)\,\Big|\int_{t_0}^{t} L(s)e^{c\left|\int_{t_0}^{s+\gamma(s)} L(u)\mathrm{d}u\right|}\mathrm{d}s\Big|^*(t_0)+$$
$$+cq+\|x\|\,E^{-1}(t)r+$$
$$+\|x\|E^{-1}(t)\,\Big|\int_{t_0}^{t} L(s)e^{c\left|\int_{t_0}^{s+\gamma(s)} L(u)\mathrm{d}u\right|}\mathrm{d}s\Big|+E^{-1}(t)\leqslant$$
$$\leqslant \big(|\bar{y}_0|+q\,\|x\|+qm\,\|x\|+cq+\|x\|\,r+$$
$$+mr\,\|x\|+1\big)E^{-1}(t)\leqslant pE^{-1}(t).$$

From the last inequality we obtain relation (4.23). Let $\bar{y} \in \mathscr{B}^*$, $\bar{z} \in \mathscr{B}^*$ and let $y = T(\bar{y})$, $z = T(\bar{z})$. Then similarly to (4.25) we get

$$
\begin{aligned}
|y-z|^*(t_0)+\operatorname{var}_{t_0}^t(y-z)(s) &\leqslant \Big|\int_{t_0}^{t} |F(\bar{y}, \bar{y}(h))-F(\bar{z}, \bar{z}(h))|(s)\mathrm{d}s\Big|^*(t_0)+ \\
&\quad + \operatorname{var}_{t_0}^t\Big(\int_{t_0}^{s} |F(\bar{y}, \bar{y}(h))-F(\bar{z}, \bar{z}(h))|(\tau)\mathrm{d}\tau \leqslant \\
&\leqslant \Big|\int_{t_0}^{t} (L|\bar{y}-\bar{z}|)(s)\mathrm{d}s\Big|^*(t_0)+ \\
&\quad + \Big|\int_{t_0}^{t} L|\bar{y}(h)-\bar{z}(h)|)(s)\mathrm{d}s\Big|^*(t_0)+ \\
&\quad + \operatorname{var}_{t_0}^t\Big(\int_{t_0}^{s} (L|\bar{y}-\bar{z}|)(\tau)\mathrm{d}\tau\Big)+ \\
&\quad + \operatorname{var}_{t_0}^t\Big(\int_{t_0}^{s} (L|\bar{y}(h)-\bar{z}(h)|)(\tau)\mathrm{d}\tau\Big) \leqslant \\
&\leqslant (q\,\|\bar{y}-\bar{z}\|)+mq\,\|\bar{y}-\bar{z}\|+ \\
&\quad + r\,\|\bar{y}-\bar{z}\|+rm\,\|\bar{y}-\bar{z}\|\big)E^{-1}(t) \leqslant \alpha\,\|\bar{y}-\bar{z}\|\,E^{-1}(t).
\end{aligned}
$$

Hence

$$\|y-z\| \leqslant \alpha\,\|\bar{y}-\bar{z}\|, \qquad \alpha \in [0, 1),$$

which completes the proof of Theorem 3.3.

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