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REFINEMENTS OF SOME CLASSICAL INEQUALITIES INVOLVING SINC AND HYPERBOLIC SINC FUNCTIONS

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Abstract. Several bounds of trigonometric-exponential and hyperbolic-exponential type for sinc and hyperbolic sinc functions are presented. In an attempt to generalize the results, some known inequalities are sharpened and extended. Hyperbolic versions are also established, along with extensions.

1. Introduction

Consider the sinc function defined by $\operatorname{sinc} x = (\sin x)/x$, for $x \neq 0$ and $\operatorname{sinc} x = 1$, for $x = 0$. A hyperbolic sinc function is defined similarly. Let us now cite some inequalities for sinc and hyperbolic sinc functions pertaining to the main results of this paper. First, the classical inequalities

$$\cos\left(\frac{x}{\sqrt{3}}\right) < \frac{\sin x}{x} < \cos\left(\frac{2}{\pi}\arccos\left(\frac{2}{\pi}\right)\cdot x\right), \quad 0 < x < \frac{\pi}{2}, \tag{1.1}$$

were established by K.S.K. Iyengar, B.S. Madhava Rao and T.S. Nanjundiah in a little-known paper [9]. See also [14]. Recently, J. Sándor ([18]) offered

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a new proof to the left inequality of (1.1) and proved its hyperbolic counterpart as follows:

$$\cosh\left(\frac{x}{\sqrt{3}}\right) < \frac{\sinh x}{x}, \quad x > 0. \tag{1.2}$$

R. Klén, M. Visuri, and M. Vuorinen ([10]) found the following inequalities

$$\left[\cos\left(\frac{x}{2}\right)\right]^2 < \frac{\sin x}{x} < \left[\cos\left(\frac{x}{3}\right)\right]^3, \quad 0 < x < \frac{\pi}{2}. \tag{1.3}$$

Y. Lv, G. Wang, and Y. Chu ([11]) obtained the following:

$$\left[\cos\left(\frac{x}{2}\right)\right]^{4/3} < \frac{\sin x}{x} < \left[\cos\left(\frac{x}{2}\right)\right]^a, \quad 0 < x < \frac{\pi}{2}, \tag{1.4}$$

where $a = (\ln(\pi/2))/(\ln\sqrt{2}) \approx 1.30299$.

The left inequality of (1.4) is sharper than the corresponding left inequality of (1.3), whereas the right inequality of (1.3) is better than that of (1.4). The analogous inequality to (1.4) is the following one:

$$\left[\cosh\left(\frac{x}{2}\right)\right]^{4/3} < \frac{\sinh x}{x} < \cosh^3 x, \quad x > 0, \tag{1.5}$$

which can be seen in [15,16,20]. Exponential-type bounds for sinc and hyperbolic sinc functions were obtained by Chesneau and Bagul in [6]. They are given below. We have

$$e^{\gamma x^2} < \frac{\sin x}{x} < e^{-x^2/6}, \quad 0 < x < \frac{\pi}{2}, \tag{1.6}$$

where $\gamma = 4\ln(2/\pi)/\pi^2$, and

$$e^{\lambda x^2} < \frac{\sinh x}{x} < e^{x^2/6}, \quad 0 < x < r, \tag{1.7}$$

where $r > 0$ and $\lambda = \ln[(\sinh r)/r]/r^2$.

For different refinements, generalizations, and recent developments regarding inequalities involving the sinc and hyperbolic sinc functions, we refer the reader to [2–5,7,12,13,17,21–26]. This article aims to present new generalized bounds for sinc and hyperbolic sinc functions. Our bounds are trigonometric-exponential and hyperbolic-exponential in nature and they refine some existing bounds in the literature.

We consider the following plan: Section 2 presents some preliminaries and lemmas. The main results are given in Section 3. Section 4 ends the article with some particular cases and discussions.

2. Preliminaries and lemmas

The following power series expansions involving Bernoulli numbers can be found in [8, 1.411]:

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-1}, \quad |x| < \frac{\pi}{2}, \tag{2.1}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \quad |x| < \pi, \tag{2.2}$$

and

$$\tanh x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}x^{2n-1}, \quad |x| < \frac{\pi}{2}, \tag{2.3}$$

where B_{2n} are the even indexed Bernoulli numbers.

We will also use the following l'Hôpital's rule of monotonicity.

Lemma 1 (l'Hôpital's rule of monotonicity [1]). *Let $f, g\colon [a,b] \longrightarrow \mathbb{R}$ be two continuous functions which are differentiable on (a,b) and $g' \neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $(f(x) - f(a))/(g(x) - g(a))$ and $(f(x) - f(b))/(g(x) - g(b))$ are also increasing (or decreasing) on (a,b). If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2 ([2, Lemma 4]). *For $x > 0$, the function*

$$k(x) = \frac{\sinh x - x\cosh x}{x^2 \sinh x}$$

is strictly increasing.

Additionally, we prove the following auxiliary results which can be of independent interest.

Lemma 3 (Sharp upper bound for hyperbolic sinc). *For $x \neq 0$, it is true that*

$$\frac{\sinh x}{x} < \sqrt{\frac{(x+\sinh x)(1+\cosh x)}{4x}}. \tag{2.4}$$

Proof. Due to the symmetry of the functions involved at both sides, it suffices to prove (2.4) for $x > 0$. We first consider

$$f(x) = x^2 + x\sinh x - 4\cosh x + 4.$$

Then

$$f'(x) = 2x + x\cosh x - 3\sinh x > 0,$$

due to well-known Cusa–Huygens inequality ([2]). Therefore, $f(x)$ is increasing for $x > 0$ and we get $f(x) > f(0)$, i.e.,

$$x^2 + x\sinh x - 4(\cosh x - 1) > 0,$$

which can be written as

$$4(\cosh x - 1)(\cosh x + 1) < x(x+\sinh x)(1+\cosh x)$$

or

$$4\sinh^2 x < x(x+\sinh x)(1+\cosh x).$$

This gives the required inequality (2.4). □

Remark 1. For $x \neq 0$, it is not difficult to prove

$$\sqrt{\frac{(x+\sinh x)(1+\cosh x)}{4x}} < \frac{2+\cosh x}{3}.$$

Thus, we have

$$\frac{\sinh x}{x} < \sqrt{\frac{(x+\sinh x)(1+\cosh x)}{4x}} < \frac{2+\cosh x}{3}, \quad x \neq 0. \tag{2.5}$$

Remark 2. A double inequality analogous to (2.5) also holds in the case of trigonometric functions. It is stated as

$$\frac{\sin x}{x} < \sqrt{\frac{(x+\sin x)(1+\cos x)}{4x}} < \frac{2+\cos x}{3}, \quad x \neq 0. \tag{2.6}$$

We skip the proof of (2.6) because it is very similar to that of (2.5).

Lemma 4 (Refined lower bound for Wilker-type inequality). *For $x \neq 0$, it is true that*

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2 + \frac{x(\cosh x - 1)(\sinh x - x)}{2\sinh^2 x} \tag{2.7}$$
$$= 2 + \frac{x(\sinh x - x)}{2(1+\cosh x)} > 2.$$

Proof. It is enough to prove that

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2 + \frac{x(\cosh x - 1)(\sinh x - x)}{2\sinh^2 x}.$$

Equivalently, it corresponds to

$$2x^2 + 2x\sinh x\cosh x > 4\sinh^2 x + x(\cosh x - 1)(\sinh x - x)$$

or

$$x^2 + x\sinh x\cosh x - 4\sinh^2 x + x\sinh x + x^2\cosh x > 0,$$

i.e.,

$$x(x + \sinh x)(1+\cosh x) > 4\sinh^2 x,$$

which is true by Lemma 3. □

Remark 3. The inequality (2.7) is a refinement of the Wilker-type inequality for hyperbolic functions established by Wu and Debnath ([19]).

Remark 4. It is interesting to see that the circular counterpart of (2.7) is also true for all non-zero real numbers. It is stated as follows:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 + \frac{x(\cos x - 1)(\sin x - x)}{2\sin^2 x} \tag{2.8}$$
$$= 2 + \frac{x(x - \sin x)}{2(1+\cos x)} > 2, \quad x \neq 0.$$

The proof of (2.8) is quite similar to that of (2.7). The importance of (2.7) lies in the fact that it is the sharpest Wilker-type inequality of its kind so far in the literature, and it holds for all non-zero real numbers, although its sharpness can be observed in $(0,\pi)$ and $(-\pi,0)$ only.

LEMMA 5. *For $x \neq 0$, it is true that*

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2 + \frac{x[\cosh(2x/p) - 1][\sinh(2x/p) - 2x/p]}{p \cdot \sinh^2(2x/p)}$$

if $p \geq 2$.

PROOF. Let

$$g(x) = 2 + x \tanh x - (x \operatorname{sech} x)^2.$$

Differentiation yields

$$\begin{aligned} g'(x) &= \tanh x + 2x^2 \operatorname{sech}^2 x \tanh x - x \operatorname{sech}^2 x \\ &= \frac{x}{\cosh x}\left(\frac{\sinh x}{x} - \frac{1}{\cosh x}\right) + 2x^2 \operatorname{sech}^2 x \tanh x > 0. \end{aligned}$$

Hence, $g(x)$ is increasing and we have that $g(x/2) \geq g(x/p)$ if $x/2 \geq x/p$, i.e., $p \geq 2$. Now, we have

$$\begin{aligned} g\left(\frac{x}{2}\right) &= 2 + \frac{x}{2}\tanh\left(\frac{x}{2}\right) - \left(\frac{x}{2}\operatorname{sech}\left(\frac{x}{2}\right)\right)^2 \\ &= 2 - \frac{x^2}{4\cosh^2(x/2)} + \frac{x\sinh(x/2)}{2\cosh(x/2)} \\ &= 2 - \frac{x^2}{2(1+\cosh x)} + \frac{x \sinh x}{2(1+\cosh x)} \\ &= 2 + \frac{x(\sinh x - x)}{2(1+\cosh x)} \\ &= 2 + \frac{x(\cosh x - 1)(\sinh x - x)}{2\sinh^2 x}. \end{aligned}$$

Similarly, we establish that

$$g\left(\frac{x}{p}\right) = 2 + \frac{x[\cosh(2x/p) - 1][\sinh(2x/p) - 2x/p]}{p \cdot \sinh^2(2x/p)}.$$

By making use of Lemma 4, the conclusion of Lemma 5 follows. □

3. Main results

We are now in a position to state and prove our main results.

THEOREM 1. *For $p > 1$, we define $\phi_p\colon (0, \pi/2] \longrightarrow \mathbb{R}$ by*

$$\phi_p(x) = \frac{\ln\left[\frac{\sin x}{x\cos(x/p)}\right]}{x^2}.$$

Then
1. *ϕ_p is strictly increasing if $p \le \sqrt{3}$,*
2. *ϕ_p is strictly decreasing if $p \ge 2$.*

PROOF. We write

$$\phi_p(x) = \frac{\ln\left[(\sin x)/x\right] - \ln[\cos(x/p)]}{x^2} = \frac{(\phi_1)_p(x)}{\phi_2(x)},$$

where $(\phi_1)_p(x) = \ln\left[(\sin x)/x\right] - \ln[\cos(x/p)]$ and $\phi_2(x) = x^2$, with $(\phi_1)_p(0+) = 0 = \phi_2(0)$. By differentiation, we get

$$\begin{aligned}\frac{(\phi_1)'_p(x)}{\phi'_2(x)} &= \frac{1}{2p}\left[p\frac{x\cos x - \sin x}{x^2\sin x} + \frac{\tan(x/p)}{x}\right]\\ &= \frac{1}{2p}\left[\frac{\tan(x/p)}{x} + p\frac{\cot x}{x} - \frac{p}{x^2}\right].\end{aligned}$$

Utilizing (2.1) and (2.2), we obtain

$$\begin{aligned}\frac{(\phi_1)'_p(x)}{\phi'_2(x)} &= \frac{1}{2p}\left[\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{p^{2n-1}\cdot(2n)!}|B_{2n}|x^{2n-2} - \sum_{n=1}^{\infty}\frac{p\cdot 2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}\right]\\ &= \frac{1}{2p}\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}\left(\frac{2^{2n}-1}{p^{2n-1}} - p\right)|B_{2n}|x^{2n-2}.\end{aligned}$$

By Lemma 1, ϕ_p will be strictly increasing if $(2^{2n}-1)/p^{2n-1} - p > 0$, i.e., $2^{2n} - 1 > p^{2n}$ or $p < h(n) := (2^{2n} - 1)^{1/2n}$. And it is easy to show that $h(n)$ is strictly increasing for $n = 1, 2, \cdots$. This implies that $p \le \inf\{h(n) : n = 1, 2, \cdots\} = h(1) = \sqrt{3}$. Similarly, we can say that ϕ_p will be strictly decreasing if we have $(2^{2n} - 1)/p^{2n-1} - p < 0$, or $p > h(n)$. So, we get $p \ge \sup\{h(n) : n = 1, 2, \cdots\} = \lim_{n\to\infty} h(n) = 2$. This completes the proof of Theorem 1. $\square$

Next, by l'Hôpital's rule, $\phi_p(0+) = \lim_{x\to 0}\phi_p(x) = 1/(2p^2) - 1/6$, and $\phi_p(\pi/2) = (4/\pi^2)\ln\left[2/[\pi\cos(\pi/(2p))]\right]$. Hence, we immediately deduce the following corollaries:

COROLLARY 1. *If $1 < p \le \sqrt{3}$ and $0 < x \le \pi/2$, then the best possible constants α_1 and β_1 such that the inequalities*

$$\cos\left(\frac{x}{p}\right)e^{\alpha_1 x^2} < \frac{\sin x}{x} < \cos\left(\frac{x}{p}\right)e^{\beta_1 x^2}$$

hold are $1/(2p^2) - 1/6$ and $(4/\pi^2)\ln\left[2/[\pi\cos(\pi/(2p))]\right]$, respectively.

COROLLARY 2. *If $p \ge 2$ and $0 < x \le \pi/2$, then the inequalities*

$$\cos\left(\frac{x}{p}\right)e^{\beta_1 x^2} < \frac{\sin x}{x} < \cos\left(\frac{x}{p}\right)e^{\alpha_1 x^2}$$

hold with the best possible constants α_1 and β_1 which are as defined in the Corollary 1.

An analogous result involving hyperbolic functions is formulated in the following theorem.

THEOREM 2. *For $p > 0$ and $r > 0$, we define a function $\psi_p\colon (0,r) \longrightarrow \mathbb{R}$ by*

$$\psi_p(x) = \frac{\ln\left[\frac{\sinh x}{x\cosh(x/p)}\right]}{x^2}.$$

Then ψ_p is strictly decreasing if $p \ge 2$. In particular, if $p \ge 2$, then the best possible constants α_2 and β_2 such that the inequalities

$$\cosh\left(\frac{x}{p}\right)e^{\alpha_2 x^2} < \frac{\sinh x}{x} < \cosh\left(\frac{x}{p}\right)e^{\beta_2 x^2}, \quad 0 < x < r, \tag{3.1}$$

hold are $\ln\left((\sinh r)/[r\cosh(r/p)]\right)/r^2$ and $1/6 - 1/(2p^2)$, respectively.

PROOF. Set $(\psi_1)_p(x) = \ln[(\sinh x)/x] - \ln[\cosh(x/p)]$ and $\psi_2(x) = x^2$. Clearly $(\psi_1)_p(0+) = 0 = \psi_2(0)$ and $\psi_p(x) = (\psi_1)_p(x)/\psi_2(x)$. In view of using Lemma 1, we differentiate and obtain

$$\frac{(\psi_1)'_p(x)}{\psi'_2(x)} = \frac{1}{2p}\left[p\frac{\coth x}{x} - \frac{p}{x^2} - \frac{\tanh(x/p)}{x}\right] := \frac{1}{2p}(\psi_3)_p(x).$$

Then, we get

$$
\begin{aligned}
(\psi_3)'_p(x) &= -\frac{p}{x}\operatorname{cosech}^2 x - \frac{p}{x^2}\coth x + \frac{2p}{x^3} - \frac{1}{px}\operatorname{sech}^2\left(\frac{x}{p}\right) + \frac{1}{x^2}\tanh\left(\frac{x}{p}\right) \\
&= -\frac{p}{x^3}\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 + \left(\frac{x}{p}\operatorname{sech}\left(\frac{x}{p}\right)\right)^2 - \frac{x}{p}\tanh\left(\frac{x}{p}\right)\right] \\
&= -\frac{p}{x^3}\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2\right. \\
&\qquad \left. - \frac{x[\cosh(2x/p)-1][\sinh(2x/p)-2x/p]}{p\cdot\sinh^2(2x/p)}\right].
\end{aligned}
$$

By Lemma 1 and Lemma 5, we conclude that ψ_p is strictly decreasing if $p \geq 2$. Consequently,

$$
\psi_p(0+) > \psi_p(x) > \psi_p(r-), \quad 0 < x < r.
$$

The desired inequalities (3.1) follow due to the limits $\psi_p(0+) = 1/6 - 1/(2p^2)$ and $\psi_p(r-) = \ln\left[(\sinh r)/[r\cosh(r/p)]\right]/r^2$. $\square$

THEOREM 3. *For $p > 1$, we define $\varphi_p\colon (0,\pi/2] \longrightarrow \mathbb{R}$ by*

$$
\varphi_p(x) = \frac{\ln\left[\frac{\sin x}{x\cosh(x/p)}\right]}{x^2}.
$$

Then φ_p is strictly decreasing if $p \geq 2$. In particular, if $p \geq 2$, then the best possible constants α_3 and β_3 such that the inequalities

$$
\cosh\left(\frac{x}{p}\right)e^{\alpha_3 x^2} < \frac{\sin x}{x} < \cosh\left(\frac{x}{p}\right)e^{\beta_3 x^2}, \quad 0 < x \leq \frac{\pi}{2} \tag{3.2}
$$

hold are $(4/\pi^2)\ln\left[2/[\pi\cosh(\pi/(2p))]\right]$ and $-\left[1/(2p^2)+1/6\right]$, respectively.

PROOF. We begin with

$$
\varphi_p(x) = \frac{\ln(\sin x/x) - \ln(\cosh x/p)}{x^2} = \frac{(\varphi_1)_p(x)}{\varphi_2(x)},
$$

where $(\varphi_1)_p(x) = \ln[(\sin x)/x] - \ln[\cosh(x/p)]$ and $\varphi_2(x) = x^2$ with $(\varphi_1)_p(0+) = 0 = \varphi_2(0)$. Differentiation gives

$$\begin{aligned}\frac{(\varphi_1)_p'(x)}{\varphi_2'(x)} &= \frac{1}{2p}\left[p\frac{x\cos x - \sin x}{x^2 \sin x} - \frac{\tanh(x/p)}{x}\right] \\ &= \frac{1}{2p}\left[p\frac{\cot x}{x} - \frac{p}{x^2} - \frac{\tanh(x/p)}{x}\right].\end{aligned}$$

Utilizing (2.1) and (2.3), we obtain

$$\begin{aligned}\frac{(\varphi_1)_p'(x)}{\varphi_2'(x)} &= \frac{1}{2p}\left[-\sum_{n=1}^{\infty}\frac{p\cdot 2^{2n}}{(2n)!}|B_{2n}|x^{2n-2} - \sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{p^{2n-1}\cdot(2n)!}B_{2n}x^{2n-2}\right] \\ &= -\frac{1}{2p}\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}\left[p\cdot|B_{2n}| - \frac{(2^{2n}-1)}{p^{2n-1}}B_{2n}\right]x^{2n-2} \\ &:= -\frac{1}{2p}\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}a_n x^{2n-2},\end{aligned}$$

where $a_n = p\cdot|B_{2n}| - [(2^{2n}-1)/p^{2n-1}]B_{2n}$. By Lemma 1, φ_p will be strictly decreasing if $a_n > 0$. But, a_n is always positive for $B_{2n} < 0$ irrespective of p. So we consider the case when $B_{2n} > 0$. In this case, $a_n > 0$ implies that $|B_{2n}| > [(2^{2n}-1)/p^{2n}]B_{2n}$ or $p^{2n} > (2^{2n}-1)$, i.e., $p > (2^{2n}-1)^{1/(2n)} := h(n)$ and $h(n)$ being strictly increasing, we write $p \geq \sup\{h(n) : n = 1, 2, \cdots\} = \lim_{n\to\infty} h(n) = 2$. Finally, $\varphi_p(0+) > \varphi_p(x) > \varphi_p(\pi/2-)$, and the limits $\varphi_p(0+) = -\left[1/(2p^2) + 1/6\right]$ and $\varphi_p(\pi/2-) = (4/\pi^2)\ln[2/[\pi\cosh(\pi/(2p))]]$ give the inequalities (3.2). $\square$

THEOREM 4. *For $p \geq 1$ and $r < \pi p/2$, we define $\chi_p \colon (0, r) \longrightarrow \mathbb{R}$ by*

$$\chi_p(x) = \frac{\ln\left[\frac{x}{\sinh x\cos(x/p)}\right]}{x^2}.$$

Then χ_p is strictly increasing. In particular, if $p \geq 1$, then the best possible constants α_4 and β_4 such that the inequalities

$$\frac{e^{\alpha_4 x^2}}{\cos(x/p)} < \frac{\sinh x}{x} < \frac{e^{\beta_4 x^2}}{\cos(x/p)}, \quad 0 < x < r \tag{3.3}$$

hold are $-\ln[r/[(\sinh r)(\cos(r/p))]]/r^2$ and $1/6 - 1/(2p^2)$, respectively.

Proof. We have

$$\chi_p(x) = \frac{(\chi_1)_p(x) - (\chi_1)_p(0+)}{\chi_2(x) - \chi_2(0)},$$

where $(\chi_1)_p(x) = \ln(x/\sinh x) - \ln[\cos(x/p)]$ and $\chi_2(x) = x^2$ with $(\chi_1)_p(0+) = 0 = \chi_2(0)$. Differentiation yields

$$\frac{(\chi_1)'_p(x)}{\chi'_2(x)} = \frac{1}{2}\frac{\sinh x - x\cosh x}{x^2 \sinh x} + \frac{1}{2p^2}\frac{\tan(x/p)}{(x/p)},$$

which is strictly increasing because of Lemma 2 and the fact that $(\tan x)/x$ is strictly increasing in $(0, \pi/2)$. Applying Lemma 1, we conclude that χ_p is strictly increasing in $(0, r)$. Hence, $\chi_p(0+) < \chi_p(x) < \chi_p(r)$ and the desired inequalities (3.3) can be obtained from this and the limits $\chi_p(0+) = 1/(2p^2) - 1/6$ and $\chi_p(r) = \ln[r/[(\sinh r)(\cos(r/p))]]/r^2$. The proof is completed. □

4. Some particular cases

In this section, we obtain some sharp inequalities from our main results by assigning appropriate values to a parameter p therein. We list the inequalities for sinc and hyperbolic sinc function as follows.

Putting $p = \sqrt{3}$ in Corollary 1 gives

$$\cos\left(\frac{x}{\sqrt{3}}\right) < \frac{\sin x}{x} < \cos\left(\frac{x}{\sqrt{3}}\right) e^{\beta_1 x^2}, \quad 0 < x \leq \frac{\pi}{2}, \tag{4.1}$$

where $\beta_1 = (4/\pi^2)\ln\left[2/(\pi\cos(\pi/2\sqrt{3}))\right] \approx 0.013219$. This includes the left inequality of (1.1). Putting $p = 2$ in Corollary 2, we obtain

$$\cos\left(\frac{x}{2}\right) e^{\beta_1^* x^2} < \frac{\sin x}{x} < \cos\left(\frac{x}{2}\right) e^{-x^2/24}, \quad 0 < x \leq \frac{\pi}{2}, \tag{4.2}$$

where $\beta_1^* = (4/\pi^2)\ln\left(2\sqrt{2}/\pi\right) \approx -0.042558$. Lower and upper bounds of (4.2) are sharper than the corresponding lower and upper bounds of (1.1) in the intervals $(\varsigma, \pi/2]$ and $(0, \zeta)$, respectively, where $\zeta \approx 1.5204$ and $\varsigma \approx 0.4633$. An upper bound of (4.2) is sharper than that of (4.1) in $(0, \zeta_1)$, where $\zeta_1 \approx 1.5346$. The double inequality (4.2) is a complete refinement of (1.3) and (1.6) and it also refines corresponding lower and upper bound of (1.4) in the intervals $(\varsigma_1, \pi/2)$ and $(0, \zeta_2)$, respectively, where $\varsigma_1 \approx 0.705$ and $\zeta_2 \approx 1.4372$. Some of

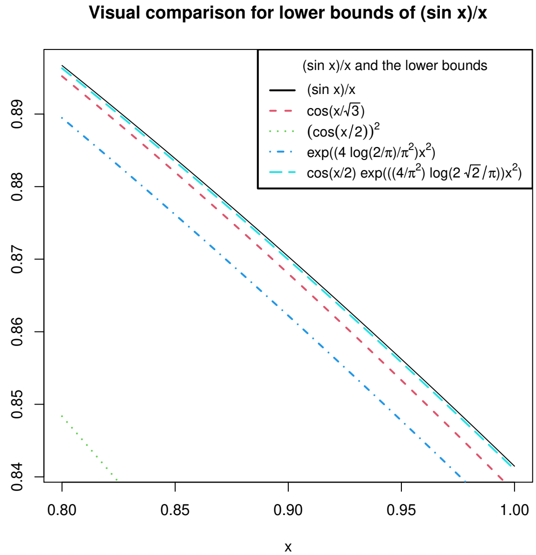

Figure 1. Visual comparison of lower bounds for $(\sin x)/x$ with $x \in [0.8, 1]$; the obtained lower bound is in lightblue color

these facts are illustrated in Figure 1 and Figure 2 for the lower and upper bounds of $(\sin x)/x$, respectively.

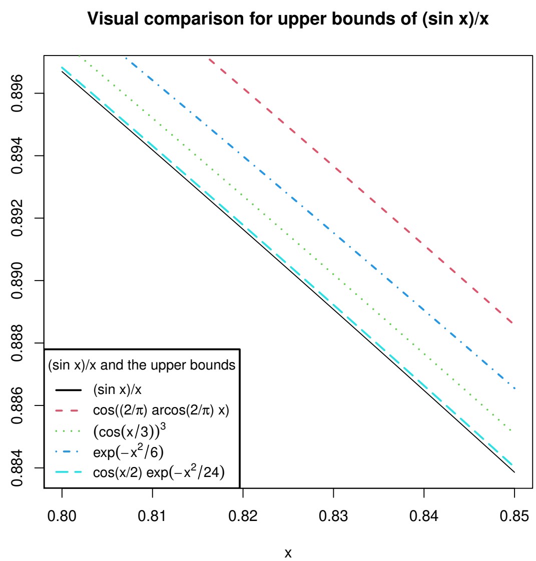

Figure 2. Visual comparison of upper bounds for $(\sin x)/x$ with $x \in [0.8, 0.85]$; the obtained upper bound is in light blue color

From Figure 1 and Figure 2, it is clear that the obtained bounds for $(\sin x)/x$ significantly improve some established bounds of the literature.

Putting $p = 2$ in Theorem 2 yields

$$\cosh\left(\frac{x}{2}\right) e^{\alpha_2 x^2} < \frac{\sinh x}{x} < \cosh\left(\frac{x}{2}\right) e^{x^2/24}, \quad 0 < x \le r, \tag{4.3}$$

where $\alpha_2 = \ln\left[(\sinh r)/(r\cosh(r/2))\right]/r^2$. The inequalities (4.3) uniformly refine (1.7). An upper bound of (4.3) is also a uniform refinement of (1.5). The lower bound of (4.3) is better than the corresponding lower bounds in (1.2) and (1.5) for smaller values of r. However, there is no strict comparison in this case for $(0, r)$. Several other inequalities can be obtained and compared with existing inequalities.

Figure 3 illustrates the sharpness of the obtained upper bound. From Figure 3, we see that the gain of the obtained upper bound in the sharpness sense is consequent.

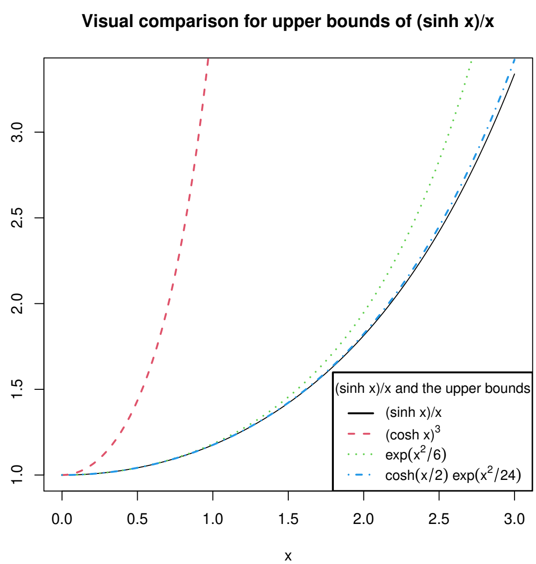

Figure 3. Visual comparison of lower bounds for $(\sinh x)/x$ with $x \in [0, 3]$; the obtained upper bound is in blue color

Note. Due to the symmetry of the functions involved all the inequalities which are true in $(0, \delta)$ are also true in $(-\delta, 0)$.

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