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GENERALIZED POLYNOMIALS ON SEMIGROUPS

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This paper is dedicated to Kazimierz Nikodem on the occasion of his 70th birthday

Abstract. This article has two main parts. In the first part we show that some of the basic theory of generalized polynomials on commutative semigroups can be extended to all semigroups. In the second part we show that if a sub-semigroup S of a group G generates G in the sense that $G = S \cdot S^{-1}$, then a generalized polynomial on S with values in an Abelian group H can be extended to a generalized polynomial on G into H. Finally there is a short discussion of the extendability of exponential functions and generalized exponential polynomials.

1. Introduction

Although there is an extensive literature about generalized polynomials on groups (see [2], [5], and their references) and commutative semigroups (see [4] and [8, sec.1-2]), it seems that very little attention has been given to non-commutative semigroups. The only instance we can find is [7], which treats generalized polynomials mapping a semigroup S satisfying the condition $gS = Sg$ for all $g \in S$ into a uniquely divisible Abelian group. We will

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show in section 2 that some of the basic theory about generalized polynomials (including the binomial theorem, the polarization formula, and the canonical representation) holds on an arbitrary semigroup if the co-domain is commutative.

Section 3 deals with the extendability of generalized polynomials. We show that if a sub-semigroup S of a group G generates G in the sense that $G = S \cdot S^{-1}$, then every generalized polynomial mapping S into an Abelian group H can be extended to a generalized polynomial from G into H. This part of the paper is based on results of Aczél, Baker, et al. [1] concerning extensions of homomorphisms; their results do not require G to be commutative.

A short final section discusses the extendability of exponentials and generalized exponential polynomials on semigroups.

Throughout the article S, T denote semigroups and G, H denote groups. We use multiplicative notation $(x, y) \mapsto xy$ for the binary operation in the domain (S or G) of our functions, where commutativity is not assumed. In the co-domain (T or H) we use additive notation $(x, y) \mapsto x + y$ or multiplicative notation depending on whether or not the binary operation is assumed to be commutative. In an Abelian group let 0 denote the identity element.

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$ let $S^n := S \times \cdots \times S$ denote the n-fold direct product. A function $h\colon S^n \to T$ is said to be an *n-homomorphism* (or in general a *multi-homomorphism*) if it is a homomorphism of S into T in each variable. We extend this definition to include $n = 0$ by defining $S^0 := S$ and the *0-homomorphisms* of S into T to be the constant functions.

If T is a commutative semigroup we use the terminology *n-additive* function for an n-homomorphism of S^n into T.

2. Generalized polynomials

Let S be a semigroup and T a commutative semigroup. For $j \in \mathbb{N}_0$ a function $f\colon S \to T$ is a *generalized monomial of degree j* if there exists a j-additive function $F\colon S^j \to T$ such that $f(x) = F(x, \dots, x)$ for all $x \in S$. Note that such f is homogeneous of degree j in the sense that for each $k \in \mathbb{N}$ and $x \in S$ we have $f(x^k) = k^j f(x)$.

For $n \in \mathbb{N}_0$ a *generalized polynomial of degree at most n* is a function $f\colon S \to T$ of the form

$$f = \sum_{j=0}^{n} f_j, \tag{2.1}$$

where $f_j\colon S \to T$ is a generalized monomial of degree j for each $j \in \{0,\ldots,n\}$. Such f has degree n if and only if $f_n \neq 0$.

For an additive function $A\colon S \to T$ we observe that $A(xy) = A(x)+A(y) = A(y) + A(x) = A(yx)$ for all $x, y \in S$, even though xy and yx may differ. In fact additive and multi-additive functions have a stronger property following from the commutativity of T. We say that a function ϕ on S is *Abelian* if $\phi(x_1 \cdots x_n) = \phi(x_{\pi(1)} \cdots x_{\pi(n)})$ for all $n \in \mathbb{N}$, all $x_1,\ldots,x_n \in S$, and all permutations π on the set $\{1,\ldots,n\}$. Every k-additive function $A\colon S^k \to T$ is Abelian in each variable, since

$$\begin{aligned}
A(s_1,\ldots,s_{j-1},&x_{\pi(1)} \cdots x_{\pi(n)}, s_j,\ldots,s_{k-1})\\
&= \sum_{i=1}^{n} A(s_1,\ldots,s_{j-1},x_{\pi(i)},s_j,\ldots,s_{k-1})\\
&= \sum_{i=1}^{n} A(s_1,\ldots,s_{j-1},x_i,s_j,\ldots,s_{k-1})\\
&= A(s_1,\ldots,s_{j-1},x_1 \cdots x_n,s_j,\ldots,s_{k-1})
\end{aligned}$$

for all $n \in \mathbb{N}$, $s_1,\ldots,s_{k-1},x_1,\ldots,x_n \in S$, and $j = 1,\ldots,k$. This property of multi-additive functions allows us to extend several standard results about generalized polynomials on commutative semigroups to arbitrary semigroups. We do this now for some well-known properties of multi-additive functions and generalized polynomials tracing back to Djoković [4].

For $n \in \mathbb{N}_0$ let $A^n(S,T)$ denote the commutative semigroup of all n-additive functions from S^n into T, so in particular $A^1(S,T) = Hom(S,T)$ is the semigroup of all homomorphisms of S into T. For any $\phi \in A^n(S,T)$ define $\phi^*\colon S \to T$ (sometimes called the diagonalization of ϕ) by

$$\phi^*(x) := \phi(x,\ldots,x), \quad x \in S.$$

We abbreviate the right hand side of the preceding equation as $\phi([x]_n)$.

For any $n \in \mathbb{N}_0$ a function ψ on S^n is said to be *symmetric* if $\psi(x_1,\ldots,x_n) = \psi(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all $x_1,\ldots,x_n \in S$ and all permutations π on the set $\{1,\ldots,n\}$ (in the case $n = 0$ every function is trivially symmetric). Let $A^n_{sym}(S,T)$ denote the sub-semigroup of all symmetric functions belonging to $A^n(S,T)$. For any $\psi \in A^n_{sym}(S,T)$ and any $x, y \in S$ let $\psi([x]_k,[y]_{n-k})$ stand for the value of ψ at any n-tuple in which k entries are x and $n-k$ entries are y. Note that if H is an Abelian group then $A^n(S,H)$ and $A^n_{sym}(S,H)$ are also Abelian groups.

If H is an Abelian group let H^S denote the set of functions mapping S into H. For each $y \in S$ let $\Delta_y \colon H^S \to H^S$ denote the right-difference operator defined by

$$\Delta_y \phi(x) := \phi(xy) - \phi(x) \quad \text{for all } x \in S,\ \phi \in H^S.$$

For compositions of such operators define

$$\Delta_{y_1,\dots,y_k} := \Delta_{y_1} \circ \cdots \circ \Delta_{y_k} \quad \text{for } y_1, \dots, y_k \in S,$$

$$\Delta_y^k := \Delta_{y,\dots,y} \quad \text{for } y \in S,\ k \in \mathbb{N}.$$

A common condition used to define a generalized polynomial f of degree at most n from a semigroup into an Abelian group is

$$\Delta_{y_1,\dots,y_{n+1}} f = 0 \quad \text{for all } y_1, \dots, y_{n+1} \in S.$$

Clearly such a definition cannot be used when the co-domain is not a group, since the difference operators cannot be defined.

The results below extend standard results on commutative semigroups, as found for example in Szekelyhidi's monograph [8], to all semigroups. We make the extensions by replacing the assumption of commutativity in S by the property that multi-additive functions are Abelian in each variable. Where the proof is essentially the same as in [8] we do not include it here. The first result is a *binomial theorem* for multi-additive symmetric functions (cf. [8, Lemma 1.2]).

Lemma 2.1. *Let S be a semigroup, T a commutative semigroup, $n \in \mathbb{N}_0$, and $\psi \in A^n_{sym}(S,T)$. Then*

$$\psi^*(xy) = \sum_{j=0}^{n} \binom{n}{j} \psi([x]_j, [y]_{n-j}), \quad x, y \in S.$$

Next is the *polarization formula* for symmetric multi-additive functions (cf. [8, Lemma 1.4]).

Lemma 2.2. *Let S be a semigroup, H an Abelian group, $n \in \mathbb{N}_0$, and $\psi \in A^n_{sym}(S,H)$. Then for all $y_1 \dots, y_k \in S$ we have*

$$\Delta_{y_1,\dots,y_k} \psi^* = \begin{cases} 0 & \text{for } k > n, \\ n!\,\psi(y_1, \dots, y_n) & \text{for } k = n. \end{cases}$$

The following corollary is immediate.

Corollary 2.3. *Let S be a semigroup, H an Abelian group, $n \in \mathbb{N}_0$, and $\psi \in A^n_{sym}(S,H)$. Then*

$$\Delta^n_y \psi^* = n!\psi^*(y), \quad y \in S.$$

A semigroup S is said to be *divisible by* $k \in \mathbb{N}$ if for every $y \in S$ there exists $x \in S$ such that $x^k = y$, and S is *uniquely divisible by* k if such x is unique for each y. The next result is an extension of [8, Lemma 1.6].

Lemma 2.4. *Let S be a semigroup, H an Abelian group, $n \in \mathbb{N}$, and $\psi \in A^n_{sym}(S,H)$. Furthermore suppose that either S is divisible by $n!$ or multiplication by $n!$ is injective in H. Then $\psi^* = 0$ only if $\psi = 0$.*

Next we discuss the uniqueness of homogeneous terms of each degree.

Lemma 2.5. *Let S be a semigroup, H an Abelian group, $n \in \mathbb{N}_0$, and suppose the functions $h_j \in A^j_{sym}(S,H)$ for $0 \le j \le n$ satisfy*

$$\sum_{j=0}^{n} h_j^* = 0. \tag{2.2}$$

If either S is divisible by $n!$ or multiplication by $n!$ is injective in H, then $h_j^* = 0$ for each $j \in \{0,\dots,n\}$.

Proof. We prove the statement by induction on n. It is obvious for $n = 0$. For $n = 1$ we have $0 = h_0^* + h_1^* = h_0 + h_1$. Thus for all $x, y \in S$ we have

$$0 = h_0 + h_1(xy) = h_0 + h_1(x) + h_1(y) = h_1(y),$$

so $0 = h_1 = h_1^*$ and therefore $0 = h_0 = h_0^*$.

Now suppose that for some $N \ge 2$ the statement is true for all $0 \le n < N$, and let

$$\sum_{j=0}^{N} h_j^* = 0. \tag{2.3}$$

By the binomial theorem (Lemma 2.1) we get for all $x, y \in S$ that

$$0 = \sum_{j=0}^{N} h_j^*(xy) = \sum_{j=0}^{N}\sum_{k=0}^{j} \binom{j}{k} h_j([x]_k, [y]_{j-k}).$$

Thus we have

$$\tag{2.4} \begin{aligned} 0 &= \sum_{j=0}^{N} [h_j^*(xy) - h_j^*(x) - h_j^*(y)] \\ &= -h_0 + \sum_{j=2}^{N} \sum_{k=1}^{j-1} \binom{j}{k} h_j([x]_k, [y]_{j-k}). \end{aligned}$$

For each fixed $y \in S$, $j \in \{2, \ldots, N\}$, and $k \in \{1, \ldots, j-1\}$, we see that

$$x \mapsto h_j([x]_k, [y]_{j-k}) \in A^k_{sym}(S, H).$$

Therefore we can view (2.4) as an equation of the form (2.2) in terms of x for each fixed y. Moreover the highest degree (in x) homogeneous term of (2.4) is $Nh_N([x]_{N-1}, y)$, which has degree $N-1$. By the inductive hypothesis each homogenous term of (2.4) vanishes identically. In particular we have

$$x \mapsto Nh_N([x]_{N-1}, y) = 0 \quad \text{for each } y \in S.$$

Putting $y = x$ here we get $Nh_N^* = 0$. If multiplication by $N!$ is injective in H then $h_N^* = 0$ follows immediately. If S is divisible by $N!$ then for each $u \in S$ we can choose $x \in S$ such that $u = x^N$, so $0 = Nh_N^*(x) = h_N^*(x^N) = h_N^*(u)$ for all $u \in S$. So in either case we have $h_N^* = 0$, and the rest follows from (2.3) by the inductive hypothesis. □

The following, which introduces the *canonical representation* (2.5) of a generalized polynomial of degree at most n, is an extension of [8, Theorem 2.3].

Theorem 2.6. *Let S be a semigroup, H an Abelian group, $n \in \mathbb{N}_0$, and $f\colon S \to H$ a generalized polynomial of degree at most n. Suppose that either multiplication by $n!$ is bijective in H, or S is uniquely divisible by $n!$. Then there exist unique functions $h_j \in A^j_{sym}(S, H)$ for $0 \le j \le n$ such that*

$$\tag{2.5} f = \sum_{j=0}^{n} h_j^*.$$

Proof. By definition we have $f = \sum_{j=0}^{n} f_j^*$ where $f_j \in A^j(S, H)$ for $0 \le j \le n$. If multiplication by $n!$ is bijective in H, then for each $j \in \{1, \ldots, n\}$ we define $h_j \in A^j_{sym}(S, H)$ by

$$h_j(x_1, \ldots, x_j) := \frac{1}{j!} \sum_{\pi} f_j(x_{\pi(1)}, \ldots, x_{\pi(j)}), \quad x_1, \ldots, x_j \in S,$$

where π runs through all permutations of the set $\{1,\ldots,j\}$. Define $h_0 := f_0$. Then we have $h_j^* = f_j^*$ for each j, and representation (2.5) holds.

On the other hand if S is uniquely divisible by $n!$ then we define

$$h_j(x_1,\ldots,x_j) := \sum_{\pi} f_j(x_{\pi(1)}^{1/j!}, x_{\pi(2)},\ldots,x_{\pi(j)}), \quad x_1,\ldots,x_j \in S,$$

for $1 \le j \le n$, and $h_0 := f_0$. It is easily verified that $h_j \in A^j_{sym}(S,H)$ and $h_j^* = f_j^*$ for each j, so again we have (2.5).

To prove uniqueness suppose there exists another representation

$$f = \sum_{j=0}^{n} g_j^*,$$

where $g_j \in A^j_{sym}(S,H)$ for $0 \le j \le n$. Then

$$0 = f - f = \sum_{j=0}^{n}(h_j^* - g_j^*) = \sum_{j=0}^{n}(h_j - g_j)^*.$$

From Lemma 2.5 we get $(g_j - h_j)^* = 0$ for each $j \in \{0,\ldots,n\}$, since $g_j - h_j \in A^j_{sym}(S,H)$. Then by Lemma 2.4 we have $g_j = h_j$ for all j. □

3. Extensions of generalized polynomials

In this section we start with the following, which is [1, Theorem 3].

Proposition 3.1. *Let G, H be groups, let S be a sub-semigroup of G such that*

$$G = S \cdot S^{-1} := \{xy^{-1} \mid x, y \in S\}, \tag{3.1}$$

and let $\psi\colon S \to H$ be a homomorphism of S into H. Then ψ can be extended to a homomorphism $\tilde{\psi}\colon G \to H$ in a unique way.

We will say that the sub-semigroup $S \subseteq G$ *generates* the group G if (3.1) holds (although that is only one very special way that S can generate G if G is not Abelian, cf. [1]).

A standard example of a semigroup generating a group is that every commutative cancellative semigroup S can be embedded in an Abelian group G, so we can view S as a sub-semigroup of G via the canonical embedding (see [3, p. 31] or [6, Theorem 3.10]). The procedure used for the embedding is similar to the construction of the field of fractions of an integral domain.

The next result generalizes Proposition 3.1 to multi-homomorphisms.

THEOREM 3.2. *Let G, H be groups, let S be a semigroup that generates G, and let $n \in \mathbb{N}_0$. Then every n-homomorphism $h\colon S^n \to H$ can be extended to an n-homomorphism $\tilde{h}\colon G^n \to H$ in a unique way.*

PROOF. For $n = 0$ this is trivial, and for $n = 1$ this is Proposition 3.1. Now let $n \geq 2$ and let $h\colon S^n \to H$ be n-homomorphic. For fixed $s_2, \ldots, s_n \in S$, consider the homomorphism

$$S \ni s \mapsto h(s, s_2, \ldots, s_n)$$

of S into H. By Proposition 3.1 there exists a unique extension of this mapping to a homomorphism of G into H. That is, there exists a unique function $h_1\colon G \times S^{n-1} \to H$ such that

$$h_1(s, s_2, \ldots, s_n) = h(s, s_2, \ldots, s_n), \quad s, s_2, \ldots, s_n \in S,$$

and h_1 is homomorphic in each variable. Now fix $g_1 \in G$ and $s_3, \ldots, s_n \in S$ (if $n \geq 3$) and consider the homomorphism

$$s \mapsto h_1(g_1, s, s_3, \ldots, s_n)$$

of S into H. Applying Proposition 3.1 again we get a unique extension of this mapping to a homomorphism of G into H, so we have a unique $h_2\colon G^2 \times S^{n-2} \to H$ which is homomorphic in each variable and agrees with h_1 (and therefore h) on S^n. Continuing this process we arrive at a unique n-homomorphism $h_n\colon G^n \to H$ such that the restriction of h_n to S^n is h. Let $\tilde{h} := h_n$. $\square$

For generalized polynomials we assume that the target space is commutative.

THEOREM 3.3. *Let G be a group, S a semigroup that generates G, and H an Abelian group. Then every generalized polynomial $f\colon S \to H$ of degree at most $n \in \mathbb{N}_0$ can be extended to a generalized polynomial $\tilde{f}\colon G \to H$ of degree at most n.*

Furthermore, if either S is uniquely divisible by $n!$ or multiplication by $n!$ is bijective in H then the extension is unique.

Proof. Let the generalized polynomial $f\colon S \to H$ have representation (2.1) with generalized monomials $f_j\colon S \to H$ of degree j for $0 \le j \le n$. By Theorem 3.2 every j-additive function $F\colon S^j \to H$ extends to a j-additive $\tilde{F}\colon G^j \to H$ in a unique way. It follows that each generalized monomial f_j extends to a generalized monomial $\tilde{f}_j\colon G \to H$, so f has an extension to a generalized polynomial $\tilde{f}\colon G \to H$ given by

$$\tilde{f} := \sum_{j=0}^{n} \tilde{f}_j.$$

The uniqueness statement follows from Theorem 2.6. $\square$

One consequence is that generalized polynomials on cancellative commutative semigroups can be extended to the groups they generate.

Corollary 3.4. *Let S be a cancellative commutative semigroup and H an Abelian group. Then every generalized polynomial $f\colon S \to H$ of degree at most $n \in \mathbb{N}_0$ can be extended to a generalized polynomial $\tilde{f}\colon G \to H$ of degree at most n, where $G = S - S$ is the (Abelian) group generated by S.*

Proof. Let $\phi\colon S \to G$ be the canonical embedding of S in G. Then we can identify S with its image $\phi(S)$ contained in G, so $S \subseteq G$. Now apply Theorem 3.3. $\square$

4. Exponentials and generalized exponential polynomials

Let S be a semigroup and R a (not necessarily commutative) ring. A function $m\colon S \to R$ is *multiplicative* if $m(xy) = m(x)m(y)$ for all $x, y \in S$. That is, m is a homomorphism of S into the multiplicative semigroup of R. A multiplicative function $m \neq 0$ is called an *exponential*. Exponentials on a semigroup are not extendable in the same way that generalized polynomials are, but we can make them so by imposing an extra condition.

A multiplicative function from a group G into a ring R must be identically 0 if it takes the value 0 at any point. Indeed, if $m(y_0) = 0$ for any $y_0 \in G$ then for all $x \in G$ we have $m(x) = m(xy_0^{-1}y_0) = m(xy_0^{-1})m(y_0) = 0$. It follows that any exponential from G into R is a multiplicative function from G into the multiplicative sub-semigroup $R^\times := R \setminus \{0\}$.

The next example shows that if a semigroup S generates a group G then, unlike our findings for generalized polynomials, an exponential on S may not extend to an exponential on G.

Example 4.1. Let $G = (\mathbb{R}_+, \cdot)$ be the multiplicative group of positive real numbers, and let R be a ring with multiplicative identity $1_R \neq 0_R$. Let S be the sub-semigroup $(0,1] \subseteq G$, and define $m\colon S \to R$ by

$$m(x) := \begin{cases} 1_R & \text{if } x = 1, \\ 0_R & \text{if } x \in (0,1). \end{cases}$$

Then S generates G and m is an exponential on S, but it is impossible to extend m to an exponential on G.

To remedy this situation we introduce a stronger type of exponential on a semigroup. We say that a function $m\colon S \to R$ is a *strict exponential* if m is a semigroup homomorphism of S into $R^\times$.

Corollary 4.2. *Let S be a semigroup generating a group G, and let R be a division ring. Then every strict exponential on S into R can be extended to an exponential on G into R in a unique way.*

Proof. Apply Theorem 3.1 with $H = R^\times$. $\square$

Let $n \in \mathbb{N}_0$. A function $\phi\colon G \to K$ is called a *generalized exponential polynomial* of degree at most n on a group G into a field K if there exist $k \in \mathbb{N}$, exponentials $m_j\colon G \to K$ and generalized polynomials $p_j\colon G \to K$ of degree at most n for $1 \leq j \leq k$, such that

$$\phi = \sum_{j=1}^{k} p_j m_j.$$

A *generalized strict exponential polynomial* of degree at most n on a semigroup S into K is defined similarly but with the additional requirement that each $m_j\colon S \to K$ is a strict exponential (and with the obvious change of G to S). The following consequence of Corollary 4.2 and Theorem 3.3 is obvious.

Corollary 4.3. *Let G be a group, S a sub-semigroup that generates G, and K a field. Then every generalized strict exponential polynomial $\phi\colon S \to K$ of degree at most n can be extended to a generalized exponential polynomial $\tilde{\phi}\colon G \to K$ of degree at most n.*

We note that Székelyhidi [9, Theorem 5] proved a similar result for the case that G is an Abelian group, S is a subgroup of G, and $K = \mathbb{C}$.

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