Annales Mathematicae Silesianae **38** (2024), no. 1, 120–133
DOI: 10.2478/amsil-2023-0028

ADAPTIVE INTEGRATION OF CONVEX FUNCTIONS OF ONE REAL VARIABLE

Szymon Wąsowicz

Dedicated to Professor Kazimierz Nikodem on the occasion of his 70th birthday

Abstract. We present an adaptive method of approximate integration of convex (as well as concave) functions based on a certain refinement of the celebrated Hermite–Hadamard inequality. Numerical experiments are performed and the role of harmonic numbers is shown.

1. Introduction

A lot of work has been done in the field of the approximate integration. Error bounds of quadratures are derived under regularity conditions of many kinds imposed on an integrands. Adaptive methods are also still being developed. In this present paper we return to the idea of the so-called *stopping inequalities*. We developed it for 3-convex functions individually in [5] or in the cooperation with Komisarski in [3]. It should be noticed that the precursors in the field were Clenshaw and Curtis [1] and Rowland and Varol [4]. Now we turn into direction of ordinary convex functions, whose concept is

Received: 14.07.2023. Accepted: 18.12.2023. Published online: 10.01.2024.
(2020) Mathematics Subject Classification: 41A55, 65D30, 65D32, 26A51, 26D15.
Key words and phrases: adaptive integration, convex function, Hermite–Hadamard inequality, quadrature, midpoint rule, trapezoid rule.

without the doubt one of the most classic ones in the whole of mathematics. In Section 2 we propose an estimate of an approximation of an integral of a convex function (see Proposition 1), which is based on a certain refinement of the celebrated Hermite–Hadamard inequality. One of the research tools is the Peano kernel. We apply our result in Section 3 to obtain the stopping inequality (7) (which is also valid for concave functions, see Corollary 3). We show its use for three functions which appear in many applications. In the last Section 4 we demonstrate the role harmonic numbers play in the approximate integration. We use them to determine the number of subdivisions of the interval of integration needed to obtain an accurate enough approximation of an integral of a reciprocal function. It is done directly and also by the use of the asymptotics of harmonic numbers. Our computations are performed by the computer programs created by us in the **R** programming language, which were run on the author's home computer equipped with fourth generation Intel® Core™ i5 processor. In tables 1 and 3 we use the accuracies $10^{-1}, 10^{-2}, \ldots, 10^{-16}$. The higher precision is impossible to achieve on **R**, because it is equipped with 16-digit floating-point arithmetic. The results of our computations are presented in three tables.

2. Certain estimate of an approximation of an integral of a convex function

Let $f\colon I \to \mathbb{R}$ be a convex function defined on a real interval I. For any $x, y \in I$ the celebrated *Hermite–Hadamard inequality* holds:

$$f\Big(\frac{x+y}{2}\Big) \le \frac{1}{y-x}\int_x^y f(t)\,\mathrm{d}t \le \frac{f(x)+f(y)}{2}.$$

Assuming that $x < y$ and multiplying it by $y - x$ we get

$$(1) \qquad (y-x) f\Big(\frac{x+y}{2}\Big) \le \int_x^y f(t)\,\mathrm{d}t \le (y-x)\frac{f(x)+f(y)}{2}.$$

A term on the left hand side is called the *simple midpoint rule*, while the term on the right hand side is called the *simple trapezoid rule*. We now use the inequality on the right first for x and $\frac{x+y}{2}$, and then for $\frac{x+y}{2}$ and y. Summing the obtained inequalities side by side and having in mind the left inequality of (1) we arrive at

$$(y-x)f\Big(\frac{x+y}{2}\Big) \le \int_x^y f(t)\,\mathrm{d}t \le \frac{1}{2}\Big[(y-x)\frac{f(x)+f(y)}{2} + (y-x)f\Big(\frac{x+y}{2}\Big)\Big]$$

for any $x,y\in I$ such that $x<y$. Denote

$$\begin{aligned}\mathcal{M}[f;x,y] &= (y-x)f\Big(\frac{x+y}{2}\Big),\\ \mathcal{T}[f;x,y] &= (y-x)\frac{f(x)+f(y)}{2},\\ \mathcal{I}[f;x,y] &= \int_x^y f(t)\,\mathrm{d}t.\end{aligned} \tag{2}$$

Then the inequality above reads as

$$\mathcal{M}[f;x,y] \le \mathcal{I}[f;x,y] \le \frac{\mathcal{M}[f;x,y]+\mathcal{T}[f;x,y]}{2}. \tag{3}$$

This is a starting point for our research. We observe that this inequality tells us that the integral of a convex function of one real variable is closer to the midpoint rule $\mathcal{M}[f;x,y]$ than to the trapezoid rule $\mathcal{T}[f;x,y]$. Moreover, the arithmetic mean on the right is a better approximation of the integral than $\mathcal{T}[f;x,y]$. After a bit of school algebra from (3) we derive the following estimate of an approximation of the integral of a convex function.

Proposition 1. *Let $I\subset\mathbb{R}$ be an interval and $f\colon I\to\mathbb{R}$ be a convex function. Then*

$$\left|\mathcal{I}[f;x,y] - \frac{3\mathcal{M}[f;x,y]+\mathcal{T}[f;x,y]}{4}\right| \le \frac{\mathcal{T}[f;x,y]-\mathcal{M}[f;x,y]}{4} \tag{4}$$

for any $x,y\in I$ such that $x<y$.

Indeed, the above inequality is trivially equivalent to (3).

Now we could ask whether or not the arithmetic mean of $\mathcal{M}[f;x,y]$ and $\frac{\mathcal{M}[f;x,y]+\mathcal{T}[f;x,y]}{2}$, i.e. the term $\frac{3\mathcal{M}[f;x,y]+\mathcal{T}[f;x,y]}{4}$ appearing in (4) is better (lower or upper) bound of the integral $\mathcal{I}[f;x,y]$. So, we investigate which of the inequalities

$$\mathcal{I}[f;x,y] \le \frac{3\mathcal{M}[f;x,y]+\mathcal{T}[f;x,y]}{4} \quad\text{or}\quad \frac{3\mathcal{M}[f;x,y]+\mathcal{T}[f;x,y]}{4} \le \mathcal{I}[f;x,y] \tag{5}$$

has a chance to hold for any convex function $f\colon I \to \mathbb{R}$ and for any $x, y \in I$ such that $x < y$. Below we show that neither of them is true, so the inequality (4) constitutes the best possibility to approximate the integral in the described context.

To find the desired counterexamples we deal with $\mathcal{I}[f;-1,1]$ and $\frac{3\mathcal{M}[f;-1,1]+\mathcal{T}[f;-1,1]}{4}$. In fact, we find the Peano kernel of the operator

$$E[f] := \mathcal{I}[f;-1,1] - \frac{3\mathcal{M}[f;-1,1]+\mathcal{T}[f;-1,1]}{4},$$

which is the function

$$K(c) = E[(\cdot - c)_+]$$

for $c \in [-1,1]$. However, further properties of the Peano kernel are not needed in this paper. We would like to emphasize that the Peano kernel is an excellent tool both to prove (this thread is absent in our paper) and to disprove the linear inequalities of the certain kind. All functions $f(x) = (x-c)_+$ are of course convex. So, if we find c_1 and c_2 such that $K(c_1) > 0$ and $K(c_2) < 0$, we will disprove both inequalities of (5). Let us compute $K(c)$.

Following the notations from (2) we have

$$\mathcal{M}[f;-1,1] = 2f(0), \quad \mathcal{T}[f;-1,1] = f(-1)+f(1), \quad \mathcal{I}[f;-1,1] = \int_{-1}^{1} f(t)\,\mathrm{d}t.$$

To construct our counterexample it is enough to determine $K(c)$ for $0 \le c \le 1$ only (to tell the truth it is not so hard to prove that K is an even function). Then for $f(x) = (x-c)_+ = \max\{x-c, 0\}$ the operators above have the form

$$\begin{aligned}
&\mathcal{M}[f;-1,1] = 2(-c)_+ = 0,\\
&\mathcal{T}[f;-1,1] = f(-1)+f(1) = (-1-c)_+ + (1-c)_+ = 1-c,\\
&\mathcal{I}[f;-1,1] = \int_{-1}^{1} (t-c)_+\,\mathrm{d}t = \int_{c}^{1} (t-c)\,\mathrm{d}t = \frac{(1-c)^2}{2}.
\end{aligned}$$

Taking this into account we compute

$$\begin{aligned}
E[(\cdot - c)_+] &= \mathcal{I}[(\cdot - c)_+;-1,1] - \frac{3\mathcal{M}[(\cdot - c)_+;-1,1] + \mathcal{T}[(\cdot - c)_+;-1,1]}{4}\\
&= \frac{(1-c)^2}{2} - \frac{1-c}{4} = \frac{(1-c)(1-2c)}{4}.
\end{aligned}$$

Hence, for instance, $K\left(\frac{1}{4}\right) > 0$, while $K\left(\frac{3}{4}\right) < 0$, which disproves both inequalities (5). Then an integral of a convex function can locate on any side of the quadrature operator $\frac{3\mathcal{M}[f;x,y]+\mathcal{T}[f;x,y]}{4}$.

3. Adaptive integration of convex functions and stopping inequality

In many cases approximation of the integral by a simple quadrature rule (e.g. by the simple midpoint or trapezoidal rules, like in this paper) is not accurate enough. To improve the accuracy one resorts to the compound quadratures. We divide the interval of integration, say $[a,b]$, into n subintervals of equal lengths: $a = x_0 < x_1 < \cdots < x_n = b$, where $x_k = a + \frac{k}{n}(b-a)$, $k = 0, 1, \dots, n$. Then we define the *compound midpoint rule* and *compound trapezoidal rule* by the formulae

$$\text{(6)}\quad \mathcal{M}_n[f;a,b] = \sum_{k=1}^{n} \mathcal{M}[f;x_{k-1},x_k] \quad \text{and} \quad \mathcal{T}_n[f;a,b] = \sum_{k=1}^{n} \mathcal{T}[f;x_{k-1},x_k],$$

respectively. Then, since $\mathcal{I}[f;x_{k-1},x_k] \approx \mathcal{M}[f;x_{k-1},x_k]$ and the integral is additive, we immediately get the approximation $\mathcal{I}[f;a,b] \approx \mathcal{M}_n[f;a,b]$ and, *mutatis mutandis*, $\mathcal{I}[f;a,b] \approx \mathcal{T}_n[f;a,b]$. Both compound quadratures are Riemann integral sums for an integrable function $f\colon [a,b] \to \mathbb{R}$. Then, of course,

$$\mathcal{I}[f;a,b] = \lim_{n\to\infty} \mathcal{M}_n[f;a,b] = \lim_{n\to\infty} \mathcal{T}_n[f;a,b],$$

whence

$$\mathcal{I}[f;a,b] = \lim_{n\to\infty} \frac{3\mathcal{M}_n[f;a,b] + \mathcal{T}_n[f;a,b]}{4}.$$

Now our task is to find n large enough to ensure the desired accuracy of the approximation of the integral $\mathcal{I}[f;a,b]$ by the compound quadrature $\frac{3\mathcal{M}_n[f;a,b]+\mathcal{T}_n[f;a,b]}{4}$. This can be achieved in many ways. It is possible to find the error bound of this quadrature (depending on n and on the 2nd derivative of the integrand) and prove that it tends to zero as $n \to \infty$. We propose another approach, which works for convex (or concave) functions, which are not necessarily twice differentiable. Namely, we will find the upper bound of a difference

$$\left|\mathcal{I}[f;a,b]-\frac{3\mathcal{M}_n[f;a,b]+\mathcal{T}_n[f;a,b]}{4}\right|,$$

which depends only on the values of the integrand and the quadratures $\mathcal{M}_n[f;a,b]$, $\mathcal{T}_n[f;a,b]$ and tends to zero as $n\to\infty$. Both these methods are called *adaptive*, because depending on the pre-selected accuracy of computations, the number of subdivisions of the interval of integration is selected accordingly.

THEOREM 2. *Let $f\colon[a,b]\to\mathbb{R}$ be either a convex, or a concave function and $n\in\mathbb{N}$. Then*

$$\left|\mathcal{I}[f;a,b]-\frac{3\mathcal{M}_n[f;a,b]+\mathcal{T}_n[f;a,b]}{4}\right|\le\frac{\left|\mathcal{T}_n[f;a,b]-\mathcal{M}_n[f;a,b]\right|}{4}.$$

PROOF. First we will prove the above inequality for convex functions. In this case, because

$$\mathcal{M}[f;x_{k-1},x_k]\le\mathcal{T}[f;x_{k-1},x_k],\quad k=1,\dots,n,$$

the absolute value on the right is not needed. In the subsequent steps we use the additivity of the integral, the triangle inequality and the inequality (4) of Proposition 1:

$$\begin{aligned}
&\left|\mathcal{I}[f;a,b]-\frac{3M_n[f;a,b]+T_n[f;a,b]}{4}\right|\\
&\qquad=\left|\sum_{k=1}^{n}\left(\mathcal{I}[f;x_{k-1},x_k]-\frac{3\mathcal{M}[f;x_{k-1},x_k]+\mathcal{T}[f;x_{k-1},x_k]}{4}\right)\right|\\
&\qquad\le\sum_{k=1}^{n}\left|\mathcal{I}[f;x_{k-1},x_k]-\frac{3\mathcal{M}[f;x_{k-1},x_k]+\mathcal{T}[f;x_{k-1},x_k]}{4}\right|\\
&\qquad\le\sum_{k=1}^{n}\frac{\mathcal{T}[f;x_{k-1},x_k]-\mathcal{M}[f;x_{k-1},x_k]}{4}\\
&\qquad=\frac{\mathcal{T}_n[f;a,b]-\mathcal{M}_n[f;a,b]}{4}=\frac{\left|\mathcal{T}_n[f;a,b]-\mathcal{M}_n[f;a,b]\right|}{4}.
\end{aligned}$$

Now if $f\colon [a,b] \to \mathbb{R}$ is a concave function, we use the above part of the proof for a convex function $(-f)$. Then, because the absolute values are present and the involved operators are linear, the inequality remains valid. □

The following *stopping inequality* is an immediate consequence of Theorem 2.

COROLLARY 3. *Let $f\colon [a,b] \to \mathbb{R}$ be either a convex, or a concave function, $n \in \mathbb{N}$ and $\varepsilon > 0$. If*

$$\frac{\left|\mathcal{T}_n[f;a,b] - \mathcal{M}_n[f;a,b]\right|}{4} \le \varepsilon, \quad \textit{(the stopping inequality)} \tag{7}$$

then

$$\left|\mathcal{I}[f;a,b] - \frac{3\mathcal{M}_n[f;a,b] + \mathcal{T}_n[f;a,b]}{4}\right| \le \varepsilon.$$

This corollary delivers the algorithm of the adaptive integration of convex functions as well as concave functions. Let $\varepsilon > 0$ be a pre-selected accuracy of computations. Because the left hand side of a stopping inequality (7) tends to zero as $n \to \infty$, this inequality is fulfilled by n large enough. Then we use this value of n to get the approximation

$$\mathcal{I}[f;a,b] \approx \frac{3\mathcal{M}_n[f;a,b] + \mathcal{T}_n[f;a,b]}{4}$$

with the desired accuracy ε. To find such n we gradually increase its value starting from $n = 1$ until the stopping inequality is fulfilled. Then the algorithm terminates.

Now it's time for the numerical experiments. We will deal with the integrals

$$\begin{aligned}
I &= \int_0^1 \frac{\mathrm{d}x}{x+1} = \ln 2 \approx 0.69314718,\\
J &= \int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}\right) \approx 0.85562439,\\
K &= \int_0^1 \mathrm{e}^{x^2}\,\mathrm{d}x = -\frac{i\sqrt{\pi}}{2}\,\mathrm{erf}(i) \approx 1.46265175.
\end{aligned}$$

Notice that on the interval $[0,1]$ the integrands of I and K are convex, while the integrand of J is concave. Using the **R** programming language we created the program computing the number n of subdivisions of the interval $[0,1]$ needed to achieve a desired accuracy ε. The initial value was $n = 1$. Then for the pre-selected value of ε the stopping inequality (7) was checked. If the condition was false, n was increased by 1. The program terminated when the condition (7) was true. Then the minimal value of n needed to satisfy the stopping inequality (7) was found. Below we present the results of our program.

Table 1. The numbers of subdivisions needed to approximate the integrals I, J, K with the specified accuracy ε

Number of subdivisions (n)			
Accuracy (ε)	I	J	K
10^{-1}	1	1	2
10^{-2}	2	2	5
10^{-3}	5	5	14
10^{-4}	16	14	42
10^{-5}	49	44	131
10^{-6}	154	138	413
10^{-7}	485	436	1304
10^{-8}	1531	1377	4122
10^{-9}	4842	4354	13035
10^{-10}	15310	13768	41219
10^{-11}	48413	43537	130343
10^{-12}	153094	137674	412181
10^{-13}	484123	435363	1303429
10^{-14}	1530932	1376739	4121805
10^{-15}	4841221	4353641	13034244
10^{-16}	15309651	13767406	41217190

4. Harmonic numbers in the approximate integration

It turns out that in the computation of the integral

$$\int_0^1 \frac{\mathrm{d}x}{x+1} = \ln 2$$

the harmonic numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

appear both in the quadrature approximating the integral and in the stopping inequality (7), on which we will concentrate here.

Let $f\colon [0,1]\to\mathbb{R}$. Following (2), (6) we compute

$$\mathcal{M}_n[f;0,1]=\sum_{k=1}^{n}\mathcal{M}\left[f;\frac{k-1}{n},\frac{k}{n}\right]=\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{2k-1}{2n}\right),$$

$$\mathcal{T}_n[f;0,1]=\sum_{k=1}^{n}\mathcal{T}\left[f;\frac{k-1}{n},\frac{k}{n}\right]=\frac{1}{2n}\sum_{k=1}^{n}\left[f\left(\frac{k-1}{n}\right)+f\left(\frac{k}{n}\right)\right].$$

Applying these formulae to $f(x)=\frac{1}{x+1}$ we arrive at

$$\begin{aligned}\mathcal{M}_n[f;0,1]&=\sum_{k=1}^{n}\frac{2}{2n+2k-1},\\ \mathcal{T}_n[f;0,1]&=\sum_{k=1}^{n}\left[\frac{1}{2n+2k-2}+\frac{1}{2n+2k}\right].\end{aligned}\tag{8}$$

We shall determine the expression for

$$\varepsilon(n):=\frac{\left|\mathcal{T}_n[f;a,b]-\mathcal{M}_n[f;a,b]\right|}{4},$$

where $f(x)=\frac{1}{x+1}$. This is the left hand side of the stopping inequality (7). Because f is convex on $[0,1]$, the absolute value is not needed (cf. (3)). We start our computations by taking into account the formulae (8):

$$\begin{aligned}\varepsilon(n)&=\frac{1}{4}\sum_{1\le k\le n}\left[\frac{1}{2n+2k-2}+\frac{1}{2n+2k}-\frac{2}{2n+2k-1}\right]\\ &=\frac{1}{4}\sum_{n+1\le n+k\le 2n}\left[\frac{1}{2(n+k)-2}+\frac{1}{2(n+k)}-\frac{2}{2(n+k)-1}\right]\\ &=\frac{1}{4}\sum_{n+1\le i\le 2n}\left[\frac{1}{2i-2}+\frac{1}{2i}-\frac{2}{2i-1}\right]\\ &=\frac{1}{8}\sum_{n+1\le i\le 2n}\left[\frac{1}{i-1}+\frac{1}{i}\right]-\frac{1}{2}\sum_{n+1\le i\le 2n}\frac{1}{2i-1}.\end{aligned}\tag{9}$$

Consider now the first sum only:

$$
\begin{aligned}
\sum_{n+1\le i\le 2n} \frac{1}{i-1} + \sum_{n+1\le i\le 2n} \frac{1}{i}
&= \sum_{n\le i-1\le 2n-1} \frac{1}{i-1} + \sum_{n+1\le i\le 2n} \frac{1}{i} \\
&= \sum_{n\le j\le 2n-1} \frac{1}{j} + \sum_{n+1\le j\le 2n} \frac{1}{j} \\
&= \frac{1}{n} + \sum_{n+1\le j\le 2n-1} \frac{1}{j} + \sum_{n+1\le j\le 2n-1} \frac{1}{j} + \frac{1}{2n} \\
&= \frac{3}{2n} + 2 \sum_{n+1\le j\le 2n-1} \frac{1}{j} \\
&= \frac{3}{2n} + 2\Big[\sum_{1\le j\le 2n-1} \frac{1}{j} - \sum_{1\le j\le n} \frac{1}{j}\Big] \\
&= \frac{3}{2n} + 2(H_{2n-1} - H_n).
\end{aligned}
\tag{10}
$$

Observe that if $n = 1$, the above computation produces an empty sum. Then either we accept the convention that the empty sum equals zero, or we compute the sum directly. Finally we can see that the obtained formula works also for $n = 1$.

Rearranging the second sum of (9) is a bit tricky. Below we use somewhat less formal way. For $n = 1$ this sum equals 1. Let $n \ge 2$.

$$
\begin{aligned}
&\sum_{n+1\le i\le 2n} \frac{1}{2i-1} = \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1} \\
&= \Big[\frac{1}{2n+1} + \frac{1}{2n+2} + \frac{1}{2n+3} + \frac{1}{2n+4} \cdots + \frac{1}{4n-2} + \frac{1}{4n-1}\Big] \\
&\quad - \Big[\frac{1}{2n+2} + \frac{1}{2n+4} + \cdots + \frac{1}{4n-2}\Big] \\
&= H_{4n-1} - H_{2n} - \frac{1}{2}\big(H_{2n-1} - H_n\big).
\end{aligned}
\tag{11}
$$

By putting the results of (10), (11) into (9) we obtain

$$\varepsilon(n) = \frac{1}{8}\left[\frac{3}{2n} + 2(H_{2n-1} - H_n)\right] - \frac{1}{2}\Big[H_{4n-1} - H_{2n} - \frac{1}{2}\big(H_{2n-1} - H_n\big)\Big].$$

Finally

$$\varepsilon(n) = \frac{3}{16n} + \frac{1}{2}\Big[H_{2n-1} + H_{2n} - H_{4n-1} - H_n\Big]. \tag{12}$$

Combining (8) and (12) with Corollary 3 we obtain

Corollary 4. *Let* $f(x) = \frac{1}{x+1}$, $\varepsilon > 0$. *If* $\varepsilon(n) \le \varepsilon$, *then*

$$\left|\mathcal{I}[f;0,1] - \frac{3\mathcal{M}_n[f;0,1] + \mathcal{T}_n[f;0,1]}{4}\right| \le \varepsilon,$$

where $\mathcal{M}_n[f;0,1]$, $\mathcal{T}_n[f;0,1]$ *are given by* (8) *and* $\varepsilon(n)$ *is given by* (12).

We are in a position to determine the number n of subdivisions of $[0,1]$ needed to approximate the integral

$$\int_0^1 \frac{\mathrm{d}x}{x+1} = \ln 2$$

with a pre-selected accuracy ε. We will do it directly by the application of the formula (12) and also by using the asymptotics of harmonic numbers. Let us recall the basic results going in this direction.

Due to Euler we have

$$\lim_{n\to\infty}(H_n - \ln n) = \gamma,$$

where $\gamma \approx 0.5772$ is the Euler-Masheroni constant. In his paper [6] from 1991 Young gave an elementary proof of the inequality

$$\frac{1}{2(n+1)} \le H_n - \ln n - \gamma \le \frac{1}{2n}.$$

As the author writes in the paper, this result was not introduced by him, but it was not widely known. It states that

$$H_n = \gamma + \ln n + O\left(\frac{1}{n}\right).$$

Unfortunately, as we will see soon, the asymptotics of this approximation is not satisfactory for our purposes. Nevertheless, there exists another aproximation of H_n given in 1993 by DeTemple in the paper [2]. He proved the inequality

$$\frac{1}{24(n+1)^2} \leq H_n - \ln\left(n+\frac{1}{2}\right) - \gamma \leq \frac{1}{24n^2},$$

which leads to the asymptotic formula

$$H_n = \gamma + \ln\left(n+\frac{1}{2}\right) + O\left(\frac{1}{n^2}\right),$$

which is considerably more precise from the previous one.

To find the asymptotic approximation of $\varepsilon(n)$ observe that in its definition (12) the coefficients in the parenthesis sum up to zero. Then it is enough to replace H_n in the formula (12) either by $\ln n$ for Young's result, or by $\ln\left(n+\frac{1}{2}\right)$ for DeTemple's one. Hence we arrive at two approximations:

$$\varepsilon(n) \approx \frac{3}{16n} + \frac{1}{2}\Big[\ln(2n-1) + \ln(2n) - \ln(4n-1) - \ln n\Big], \qquad \text{(Young)} \tag{13}$$

$$\varepsilon(n) \approx \frac{3}{16n} + \frac{1}{2}\left[\ln\left(2n-\frac{1}{2}\right) + \ln\left(2n+\frac{1}{2}\right) - \ln\left(4n-\frac{1}{2}\right) - \ln\left(n+\frac{1}{2}\right)\right] \qquad \text{(DeTemple).} \tag{14}$$

The first one is $O\left(\frac{1}{n}\right)$, while the second one is $O\left(\frac{1}{n^2}\right)$. Table 2 presents for selected n the values of $\varepsilon(n)$ computed by (12), the above Young's approximation and DeTemple's approximation, respectively. The last two columns contain the relative errors of these approximations expressed as percentage. The appropriate computer program was also created in **R**.

As we can see, Young's approximation (13) is completely useless for our purposes. The relative error rapidly grows as n increases. DeTemple's approximation (14), on the other hand, seems to be reasonable and the relative error keeps the acceptable constant level. Another program created in **R** computed the numbers of subdivisions needed to achieve the pre-selected accuracy $\varepsilon \in \{10^{-1}, \dots, 10^{-16}\}$. In the second column the exact formula (12) was applied to compute $\varepsilon(n)$. In the third column DeTemple approximation of $\varepsilon(n)$ given by (14) was applied.

Table 2. The values of $\varepsilon(n)$ and its approximations

		Approximation		Relative error in %	
n	$\varepsilon(n)$ by (12)	Young	DeTemple	Young	DeTemple
5	$9.32 \cdot 10^{-4}$	$4.80 \cdot 10^{-2}$	$1.25 \cdot 10^{-3}$	5048	34
10	$2.34 \cdot 10^{-4}$	$2.45 \cdot 10^{-2}$	$3.32 \cdot 10^{-4}$	10375	42
20	$5.86 \cdot 10^{-5}$	$1.24 \cdot 10^{-2}$	$8.54 \cdot 10^{-5}$	21038	46
50	$9.37 \cdot 10^{-6}$	$4.98 \cdot 10^{-3}$	$1.39 \cdot 10^{-5}$	53035	48
100	$2.34 \cdot 10^{-6}$	$2.50 \cdot 10^{-3}$	$3.50 \cdot 10^{-6}$	106368	49
200	$5.86 \cdot 10^{-7}$	$1.25 \cdot 10^{-3}$	$8.76 \cdot 10^{-7}$	213034	50
500	$9.37 \cdot 10^{-8}$	$5.00 \cdot 10^{-4}$	$1.40 \cdot 10^{-7}$	533034	50
1000	$2.34 \cdot 10^{-8}$	$2.50 \cdot 10^{-4}$	$3.51 \cdot 10^{-8}$	1066367	50
5000	$9.38 \cdot 10^{-10}$	$5.00 \cdot 10^{-5}$	$1.41 \cdot 10^{-9}$	5333031	50
10000	$2.34 \cdot 10^{-10}$	$2.50 \cdot 10^{-5}$	$3.52 \cdot 10^{-10}$	10666430	50
50000	$9.37 \cdot 10^{-12}$	$5.00 \cdot 10^{-6}$	$1.41 \cdot 10^{-11}$	53336330	50
100000	$2.34 \cdot 10^{-12}$	$2.50 \cdot 10^{-6}$	$3.51 \cdot 10^{-12}$	106686500	50

Table 3. The numbers of subdivisions needed to compute the integral $\int_0^1 \frac{\mathrm{d}x}{x+1}$ with the specified accuracy ε

Number of subdivisions (n)			
Accuracy (ε)	Exact (harmonic numbers)	DeTemple's approximation	Relative error in %
10^{-1}	1	1	0
10^{-2}	2	2	0
10^{-3}	5	6	20
10^{-4}	16	19	19
10^{-5}	49	60	22
10^{-6}	154	188	22
10^{-7}	485	593	22
10^{-8}	1531	1875	22
10^{-9}	4842	5929	22
10^{-10}	15310	18750	22
10^{-11}	48413	59291	22
10^{-12}	153089	187305	22
10^{-13}	483989	587602	21
10^{-14}	1526699	1692129	11
10^{-15}	4712203	3090396	34
10^{-16}	12272081	3540284	71

Notice that the numbers in the second column of Table 3 concerning the accuracies $\varepsilon \in \{10^{-12}, \dots, 10^{-16}\}$ differ from the analogous numbers from the second column of Table 1. This is caused by different algorithms used for computing the contents of both tables.

The last three rows of the third column of Table 3 (DeTemple's approximation) cannot be considered reliable, since the calculations were carried out at the limit of accuracy of floating point arithmetic of $\mathbf{R}$.

Also here we can see that the relative error is acceptable and, roughly speaking, constant, except for the last three rows of the Table 3. Then the properly chosen asymptotics of harmonic numbers can also be applied to estimate the accuracy of the approximation of our integral.

References

[1] C.W. Clenshaw and A.R. Curtis, *A method for numerical integration on an automatic computer*, Numer. Math. **2** (1960), 197–205.

[2] D.W. DeTemple, *A quicker convergence to Euler's constant*, Amer. Math. Monthly **100** (1993), no. 5, 468–470.

[3] A. Komisarski and S. Wąsowicz, *On optimal inequalities between three-point quadratures*, Aequationes Math. **96** (2022), no. 3, 621–638.

[4] J.H. Rowland and Y.L. Varol, *Exit criteria for Simpson's compound rule*, Math. Comp. **26** (1972), 699–703.

[5] S. Wąsowicz, *On a certain adaptive method of approximate integration and its stopping criterion*, Aequationes Math. **94** (2020), no. 5, 887–898.

[6] R.M. Young, *Euler's constant*, Math. Gaz. **75** (1991), no. 472, 187–190.

Department of Mathematics
University of Bielsko–Biala
Willowa 2
43–309 Bielsko–Biała
Poland
e-mail: swasowicz@ubb.edu.pl