Annales Mathematicae Silesianae **38** (2024), no. 1, 12–17
DOI: 10.2478/amsil-2023-0031

NOTE ON AN ITERATIVE FUNCTIONAL EQUATION

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Dedicated to Professor Kazimierz Nikodem on his seventieth birthday

Abstract. We study the problem of solvability of the equation

$$\varphi(x) = \int_\Omega g(\omega)\varphi\big(f(x,\omega)\big)P(d\omega) + F(x),$$

where P is a probability measure on a σ-algebra of subsets of Ω, assuming Hölder continuity of F on the range of f.

Fix a probability space $(\Omega, \mathcal{A}, P)$, a separable metric space (X, ρ), a separable Banach space Y over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and an $\alpha \in (0, 1]$.

Motivated by [1], [2] and [3] we consider solutions $\varphi\colon X \to Y$ of the equation

$$\varphi(x) = \int_\Omega g(\omega)\varphi\big(f(x,\omega)\big)P(d\omega) + F(x) \tag{1}$$

assuming the following hypotheses.

Received: 31.08.2023. Accepted: 21.12.2023. Published online: 10.01.2024.
(2020) Mathematics Subject Classification: 39B12.
Key words and phrases: iterative functional equations, Hölder continuous functions.
The research was supported by the Institute of Mathematics of the University of Silesia (Iterative Functional Equations and Real Analysis program).

(H_1) Function f maps $X \times \Omega$ into X and $f(x,\cdot)$ is measurable for $\mathcal{A}$ for every $x \in X$, i.e.,

$$\{\omega \in \Omega\colon f(x,\omega) \in B\} \in \mathcal{A} \quad \text{for } x \in X \text{ and Borel } B \subset X.$$

(H_2) Function $g\colon \Omega \to \mathbb{K}$ is integrable for P,

$$\int_\Omega |g(\omega)|^{\frac{1}{\alpha}} \rho\big(f(x,\omega),x\big) P(d\omega) < \infty \qquad \text{for } x \in X$$

and

$$\int_\Omega |g(\omega)|^{\frac{1}{\alpha}} \rho\big(f(x,\omega),f(z,\omega)\big) P(d\omega) \le \lambda \rho(x,z) \quad \text{for } x,z \in X$$

with a $\lambda \in (0,1)$.

For $A \subset X$ denote by $\mathcal{H}_\alpha(A)$ the family of all functions $F\colon X \to Y$ for which there is an $L \in [0,\infty)$ such that

$$\|F(x) - F(z)\| \le L\rho(x,z)^\alpha \quad \text{for } x,z \in A.$$

Integrating vector–valued functions we use the Bochner integral.

We start with the following lemma.

Lemma. *Assume* (H_1) *and* (H_2). *If* $F \in \mathcal{H}_\alpha\big(f(X \times \Omega)\big)$, *then for every* $x \in X$ *the function* $g \cdot F \circ f(x,\cdot)$ *is integrable for* P *and the function*

$$x \mapsto \int_\Omega g(\omega) F\big(f(x,\omega)\big) P(d\omega), \quad x \in X,$$

is in $\mathcal{H}_\alpha(X)$.

Proof. Fix $x \in X$. Clearly $g \cdot F \circ f(x,\cdot)$ is measurable for $\mathcal{A}$ and with arbitrarily fixed $z \in f(X \times \Omega)$ and an $L \in (0,\infty)$ by Jensen's inequality (see, e.g., [4, 10.2.6]) we have

$$\int_\Omega \|g(\omega)F\big(f(x,\omega)\big)\| P(d\omega) \le \int_\Omega |g(\omega)| \|F\big(f(x,\omega)\big) - F\big(f(z,\omega)\big)\| P(d\omega)$$

$$+ \int_\Omega |g(\omega)| \|F\big(f(z,\omega)\big) - F(z)\| P(d\omega) + \|F(z)\| \int_\Omega |g(\omega)| P(d\omega)$$

$$\leq L\Big(\int_\Omega |g(\omega)|^{\frac{1}{\alpha}}\rho\big(f(x,\omega),f(z,\omega)\big)P(d\omega)\Big)^\alpha$$

$$+ L\left(\int_\Omega |g(\omega)|^{\frac{1}{\alpha}}\rho\big(f(z,\omega),z\big)P(d\omega)\right)^\alpha + \|F(z)\|\int_\Omega |g(\omega)|P(d\omega)$$

$$\leq L\lambda^\alpha\rho(x,z)^\alpha + L\Big(\int_\Omega |g(\omega)|^{\frac{1}{\alpha}}\rho\big(f(z,\omega),z\big)P(d\omega)\Big)^\alpha$$

$$+ \|F(z)\|\int_\Omega |g(\omega)|P(d\omega) < \infty.$$

Thus $g\cdot F\circ f(x,\cdot)$ is integrable for P.

For the proof of the second part note that with an $L\in(0,\infty)$ for all $x,z\in X$ we have

$$\left\|\int_\Omega g(\omega)F\big(f(x,\omega)\big)P(d\omega) - \int_\Omega g(\omega)F\big(f(z,\omega)\big)P(d\omega)\right\|$$

$$\leq \int_\Omega |g(\omega)|\|F\big(f(x,\omega)\big) - F\big(f(z,\omega)\big)\|P(d\omega)$$

$$\leq L\Big(\int_\Omega |g(\omega)|^{\frac{1}{\alpha}}\rho\big(f(x,\omega),f(z,\omega)\big)P(d\omega)\Big)^\alpha \leq L\lambda^\alpha\rho(x,z)^\alpha. \qquad \square$$

Assuming (H_1) and (H_2) and making use of the Lemma for every $F\in\mathcal{H}_\alpha\big(f(X\times\Omega)\big)$ we define a sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{H}_\alpha(X)$ by

$$F_0(x) = F(x), \quad F_n(x) = \int_\Omega g(\omega)F_{n-1}(f(x,\omega))P(d\omega)$$

for $x\in X$ and $n\in\mathbb{N}$; moreover we put

$$\gamma = \int_\Omega g dP.$$

Our theorem reads.

Theorem. *Assume* (H_1) *and* (H_2). *Let* $F\in\mathcal{H}_\alpha\big(f(X\times\Omega)\big)$.

(i) *If* $\gamma\neq 1$, *then equation* (1) *has exactly one solution* $\varphi\in F+\mathcal{H}_\alpha(X)$.

(ii) *If* $\gamma = 1$ *and there is an* $x_0\in f(X\times\Omega)$ *such that* $\lim_{n\to\infty}F_n(x_0)=0$, *then equation* (1) *has a solution* $\varphi\in F+\mathcal{H}_\alpha(X)$ *unique up to an additive constant.*

(iii) *If* $\gamma = 1$ *and equation* (1) *has a solution* $\varphi\in F+\mathcal{H}_\alpha(X)$, *then* $\lim_{n\to\infty}F_n(x)=0$ *for every* $x\in f(X\times\Omega)$.

Proof. Put $X_0 = f(X \times \Omega)$ and consider X_0 with the metric d given by $d = (\rho|_{X_0 \times X_0})^\alpha$. Then f_0 defined as $f|_{X_0 \times \Omega}$ maps $X_0 \times \Omega$ into X_0, $f_0(x, \cdot)$ is measurable for $\mathcal{A}$ for every $x \in X_0$ and by Jensen's inequality

$$\int_\Omega |g(\omega)| d\big(f_0(x,\omega), x\big) P(d\omega) \le \left(\int_\Omega |g(\omega)|^{\frac{1}{\alpha}} \rho\big(f(x,\omega), x\big) P(d\omega)\right)^\alpha < \infty$$

for $x \in X_0$, and

$$\int_\Omega |g(\omega)| d\big(f_0(x,\omega), f_0(z,\omega)\big) P(d\omega)$$
$$\le \left(\int_\Omega |g(\omega)|^{\frac{1}{\alpha}} \rho\big(f(x,\omega), f(z,\omega)\big) P(d\omega)\right)^\alpha \le \big(\lambda\rho(x,z)\big)^\alpha = \lambda^\alpha d(x,z)$$

for $x, z \in X_0$.

We will now prove theses (i) and (ii).

It follows from [2, Theorem 2.3] in case (i) and from [3, Theorem 2.1] in case (ii) that there is a $\varphi_0 \colon X_0 \to Y$ such that

$$\|\varphi_0(x) - \varphi_0(z)\| \le L\rho(x,z)^\alpha \quad \text{for } x, z \in X_0$$

with an $L \in [0, \infty)$ and

$$\varphi_0(x) = \int_\Omega g(\omega)\varphi_0\big(f(x,\omega)\big) P(d\omega) + F(x) \quad \text{for } x \in X_0.$$

Using the Lemma define $\varphi \colon X \to Y$ by

$$\varphi(x) = \int_\Omega g(\omega)\varphi_0\big(f(x,\omega)\big) P(d\omega) + F(x)$$

and note that $\varphi \in F + \mathcal{H}_\alpha(X)$, $\varphi|_{X_0} = \varphi_0$, and φ solves (1).

To prove the uniqueness suppose that $\varphi_1, \varphi_2 \in F + \mathcal{H}_\alpha(X)$ are solutions of (1). Then φ defined as $\varphi_1 - \varphi_2$ is in $\mathcal{H}_\alpha(X)$ and solves (1) with $F = 0$. Denoting by L the smallest Lipschitz-Hölder constant for φ, for all $x, z \in X$ we have

$$\|\varphi(x) - \varphi(z)\| \le \int_\Omega |g(\omega)| \|\varphi\big(f(x,\omega)\big) - \varphi\big(f(z,\omega)\big)\| P(d\omega) \le L\lambda^\alpha \rho(x,z)^\alpha,$$

whence $L = 0$ and φ is a constant function. In case (i) the only constant solution of (1) with $F = 0$ is the zero function, whence $\varphi_1 = \varphi_2$.

To get (iii) it is enough to note that if $\varphi\colon X \to \mathbb{R}$ is a solution of (1) in $F+\mathcal{H}_\alpha(X)$, then $\varphi|_{X_0}$ is a Lipschitz solution of (1) with F replaced by $F|_{X_0}$ and to apply [3, Theorem 2.1]. □

Remark. Assume (H_1) and (H_2). If $F \in \mathcal{H}_\alpha\big(f(X\times\Omega)\big)$ and $x_0 \in X$, then each of the following two conditions implies that $\lim_{n\to\infty} F_n(x_0) = 0$:

(a) $f(x_0,\cdot) = x_0$ a.e. for P and $F(x_0) = 0$;

(b) $F \circ f(\cdot,\omega_1) \circ \ldots \circ f(\cdot,\omega_n)(x_0) = 0$ for every $n \in \mathbb{N}$ and $\omega_1,\ldots,\omega_n \in \Omega$.

Example. Assume that X is a separable normed space over $\mathbb{K}$ and let $x^* \in X^*$, $(p_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$, $(\gamma_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ with

$$\sum_{n=1}^{\infty} p_n = 1, \quad \sum_{n=1}^{\infty} p_n|\gamma_n| < \infty,$$

$$\sum_{n=1}^{\infty} p_n|\gamma_n|^{\frac{1}{\alpha}} < \infty, \quad \|x^*\|\sum_{n=1}^{\infty} p_n|\gamma_n|^{\frac{1}{\alpha}}\|a_n\| < 1, \quad \sum_{n=1}^{\infty} p_n|\gamma_n|^{\frac{1}{\alpha}}\|b_n\| < \infty.$$

Put

$$\gamma = \sum_{n=1}^{\infty} p_n\gamma_n, \quad X_0 = \operatorname{Lin}(\{a_n : n\in\mathbb{N}\}\cup\{b_n : n\in\mathbb{N}\})$$

and let $F \in \mathcal{H}_\alpha(X_0)$.

If $\gamma \neq 1$, then by part (i) of the Theorem (with $\Omega = \mathbb{N}$, $P(\{n\}) = p_n$ for $n\in\mathbb{N}$, and $f(x,n) = (x^*x)a_n + b_n$ for $(x,n) \in X\times\mathbb{N}$, $g(n) = \gamma_n$ for $n \in \mathbb{N}$) the equation

$$\varphi(x) = \sum_{n=1}^{\infty} \gamma_n p_n \varphi\big((x^*x)a_n + b_n\big) + F(x) \tag{2}$$

has exactly one solution $\varphi \in F + \mathcal{H}_\alpha(X)$; if F is also continuous, or if F is also uniformly continuous, then so is φ.

If $\gamma = 1$, $x^*b_n = 0$ and $F(b_n) = 0$ for every $n \in \mathbb{N}$, then by part (ii) of the Theorem and the Remark (cf. condition (b)) equation (2) has a solution $\varphi \in F + \mathcal{H}_\alpha(X)$ unique up to an additive constant; if F is also continuous, or if F is also uniformly continuous, then so is φ.

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