sciendo

Annales Mathematicae Silesianae **38** (2024), no. 1, 93–110
DOI: 10.2478/amsil-2024-0009

ON FUNCTIONS WITH MONOTONIC DIFFERENCES

Teresa Rajba

Dedicated to Professor Kazimierz Nikodem on his 70^{th} *birthday*

Abstract. Motivated by the Szostok problem on functions with monotonic differences (2005, 2007), we consider a-Wright convex functions as a generalization of Wright convex functions. An application of these results to obtain new proofs of known results as well as new results is presented.

1. Introduction

Let I be a subinterval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ be a function. The function f is called *Wright convex* ([11]) if

$$f(\alpha x + (1-\alpha)y) + f((1-\alpha)x + \alpha y) \le f(x) + f(y) \quad (x, y \in I,\ \alpha \in [0,1]).$$

The function f is called *strictly Wright convex* if

$$f(\alpha x + (1-\alpha)y) + f((1-\alpha)x + \alpha y) < f(x) + f(y) \quad (x, y \in I,\ \alpha \in [0,1]).$$

Received: 22.10.2023. Accepted: 06.03.2024. Published online: 20.03.2024.
(2020) Mathematics Subject Classification: 39B12, 30D05, 39B52, 39B22, 26A18.
Key words and phrases: monotonic differences, non-decreasing differences, Wright convexity.

Let $a \geq 0$ be a fixed real number. The *difference operator* of the function f has the form

$$\Delta_a f(x) = f(x+a) - f(x) \quad (x \in I \cap (I-a)).$$

According to [4], the Wright convexity can be characterized as follows.

PROPOSITION 1.1. *The function $f\colon I \to \mathbb{R}$ is Wright convex if and only if*

$$\Delta_t \Delta_a f(x) \geq 0 \quad (t, a > 0,\ x \in I \cap [I-(t+a)]). \tag{1.1}$$

For strictly Wright convex functions

$$\Delta_t \Delta_a f(x) > 0 \quad (t, a > 0,\ x \in I \cap [I-(t+a)]).$$

By (1.1), Wright convex functions can be characterized as functions f, for which the difference operators $f_a = \Delta_a f$ are non-decreasing for all $a > 0$. Similarly, f is Wright concave if the difference operators f_a are non-increasing for all $a > 0$.

There are many generalizations of Wright convex functions. The author [7] studied a generalization of Wright convex functions via randomization. The author [7] studied (among others) the non-decreasing function f that satisfies the inequality

$$\mathbb{E}\, \nabla_\theta \nabla_t f(x) \geq 0 \quad (x \in \mathbb{R},\ t > 0),$$

where $\mathbb{E}X$ is the expectation of a real valued random variable X, θ is a non-negative real valued random variable and ∇_a is the *backward difference operator* defined by $\nabla_a f(x) = f(x) - f(x-a)$ (obviously $\nabla_a f(x+a) = \Delta_a f(x)$).

T. Szostok [8, 9] posed a problem for functions f defined on an interval. Assume, that for every $a > 0$ the function f_a is strictly monotonic. Is f_a strictly increasing for every $a > 0$ or strictly decreasing for every $a > 0$? Szostok [10] proved that the answer is positive if f is continuous. Balcerowski [1] proved that the answer is positive in general.

Motivated by the Szostok problem, we consider some convexity concept as a generalization of Wright convexity of functions. Given $a \geq 0$, we say that the function $f\colon I \to \mathbb{R}$ is *a-Wright convex* if

$$\Delta_t \Delta_a f(x) \geq 0 \quad (t > 0,\ x \in I \cap [I-(t+a)]).$$

In other words, f is a-Wright convex if the difference operator f_a is non-decreasing. We say that f is *a-Wright concave* if the function f_a is non-increasing. Let S be a set such that $S \subset [0, \infty)$. We say that f is *S-Wright* *convex* (*S-Wright concave*) if f is a-Wright convex (a-Wright concave) for all $a \in S$. We put

$$A_f = \{a \geq 0 \colon f \text{ is a-Wright convex}\},$$

$$B_f = \{a \geq 0 \colon f \text{ is a-Wright concave}\}.$$

Then f is Wright convex if and only if $A_f = [0, \infty)$ and f is Wright concave if and only if $B_f = [0, \infty)$.

Let BV be the class of functions $f \colon \mathbb{R} \to \mathbb{R}$ having bounded variation over any finite interval. In this paper, we prove that the sets A_f and B_f are additive closed subsemigroups of $[0, \infty)$ containing 0, and if $S \subset [0, \infty)$ is such a semigroup, then there is a function $f \in BV$ such that $A_f = S$ ($B_f = S$). Moreover, we study relationships between the sets A_f and B_f corresponding to the function $f \in BV$. We give an application of these results to give new proofs of some known results as well as we obtain new results.

2. a-Wright convex functions

For the standard properties of difference operator, we refer to [3].

LEMMA 2.1. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. Then*

$$\Delta_{a_1+a_2} f(x) = \Delta_{a_2} f(x + a_1) + \Delta_{a_1} f(x), \tag{2.1}$$

for all $x \in \mathbb{R}$, $a_1, a_2 > 0$.

PROOF.

$$\begin{aligned} \Delta_{a_1+a_2} f(x) &= f(x + a_1 + a_2) - f(x) \\ &= (f(x + a_1 + a_2) - f(x + a_1)) + (f(x + a_1) - f(x)) \\ &= \Delta_{a_2} f(x + a_1) + \Delta_{a_1} f(x). \end{aligned}$$ □

LEMMA 2.2. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function of the form $f(x) = \int_{-\infty}^{x} g(u)du$, where $g \colon \mathbb{R} \to \mathbb{R}$ is an integrable function such that $g(x) = 0$ if $x < 0$. Then*

$$\Delta_a f(x) = \int_{-\infty}^{x} \Delta_a g(u)du, \tag{2.2}$$

for all $x \in \mathbb{R}$, $a > 0$.

Proof.

$$\begin{aligned}\Delta_a f(x) &= \Delta_a \int_{-\infty}^{x} g(u)du = \int_{-\infty}^{x+a} g(u)du - \int_{-\infty}^{x} g(u)du \\ &= \int_{-\infty}^{x} g(u+a)du - \int_{-\infty}^{x} g(u)du \\ &= \int_{-\infty}^{x} (g(u+a)-g(u))du = \int_{-\infty}^{x} \Delta_a g(u)du. \qquad \square\end{aligned}$$

Theorem 2.3. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a function such that $f\in BV$. Then A_f is an additive closed subsemigroup of $[0,\infty)$ containing* 0.

Proof. Let $f \in BV$. Obviously $0 \in A_f$. Let $a_1, a_2 \geq 0$ be such that $a_1, a_2 \in A_f$. If $a_1 = 0$ or $a_2 = 0$, then obviously $a_1 + a_2 \in A_f$. Assume that $a_1, a_2 > 0$. By (2.1), $a_1 + a_2 \in A_f$. This implies that A_f is an additive subsemigroup of $[0,\infty)$.

Assume now, that $a_1 > 0, a_2 > 0, \ldots$ be such that $a_1 \in A_f, a_2 \in A_f, \ldots$ and $\lim_{n\to\infty} a_n = a_0 \in \mathbb{R}$. Since $a_1, a_2, \ldots > 0$, it follows that $a_0 \geq 0$. If $a_0 = 0$, then obviously $a_0 \in A_f$. Assume that $a_0 > 0$. Since $f \in BV$, there exist non-decreasing functions $\varphi, \psi\colon \mathbb{R}\to\mathbb{R}$ such that $f = \varphi - \psi$.

Taking into account that non-decreasing functions are continuous λ-a.e. and $\lim_{n\to\infty} a_n = a_0$, we obtain that $\varphi(x+a_n) \xrightarrow[n\to\infty]{} \varphi(x+a_0)$ and $\psi(x+a_n) \xrightarrow[n\to\infty]{} \psi(x+a_0)$ λ-a.e., consequently, $\Delta_{a_n}\varphi(x) = \varphi(x+a_n) - \varphi(x) \xrightarrow[n\to\infty]{} \varphi(x+a_0) - \varphi(x) = \Delta_{a_0}\varphi(x)$, $\Delta_{a_n}\psi(x) = \psi(x+a_n) - \psi(x) \xrightarrow[n\to\infty]{} \psi(x+a_0) - \varphi(x) = \Delta_{a_0}\psi(x)$ λ-a.e., which implies $\Delta_{a_n} f(x) = \Delta_{a_n}\varphi(x) - \Delta_{a_n}\psi(x) \xrightarrow[n\to\infty]{} \Delta_{a_0}\varphi(x) - \Delta_{a_0}\psi(x) = \Delta_{a_0} f(x)$ λ-a.e. Taking into account that $a_1, a_2, \ldots \in A_f$, i.e. the functions $\Delta_{a_1} f, \Delta_{a_2} f, \ldots$ are non-decreasing, we obtain that $\Delta_{a_0} f$ is also non-decreasing. Indeed, contrary to our statement suppose, that $\Delta_{a_0} f$ is not non-decreasing. Then, there exist $x_1 < x_2$ such that $\Delta_{a_0} f(x_2) - \Delta_{a_0} f(x_1) < 0$. Without loss of generality, we may assume that x_1, x_2 are the points of continuity of $\Delta_{a_0} f$. Since $\Delta_{a_n} f$ is non-decreasing, it follows $\Delta_{a_n} f(x_2) - \Delta_{a_n} f(x_1) \geq 0$, $n = 1, 2, \ldots$. Consequently, we obtain

$$0 \leq \lim_{n\to\infty} \Delta_{a_n} f(x_2) - \lim_{n\to\infty} \Delta_{a_n} f(x_1) = \Delta_{a_0} f(x_2) - \Delta_{a_0} f(x_1) < 0,$$

which is a contradiction. Thus, we obtain that $\Delta_{a_0} f$ is non-decreasing, which implies that $a_0 \in A_f$. This completes the proof. $\square$

By Theorem 2.3, we obtain the following corollaries.

Corollary 2.4. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a function such that* $f \in BV$*. Then* B_f *is an additive closed subsemigroup of* $[0,\infty)$ *containing* 0.

Corollary 2.5. *If there exists a sequence of positive numbers* $a_1, a_2, \ldots \in A_f$ (B_f) *such that* $\lim_{n\to\infty} a_n = 0$*, then* $A_f = [0,\infty)$ $(B_f = [0,\infty))$.

Proof. Let $a_1 > 0, a_2 > 0, \ldots$ be such that $a_1 \in A_f, a_2 \in A_f, \ldots$ and $\lim_{n\to\infty} a_n = 0$. Then the additive semigroup, which is generated by the set $\{a_n\}_{n=1}^{\infty}$ is dense in the set $[0,\infty)$. Consequently, the closed additive semigroup which is generated by the set $\{a_n\}_{n=1}^{\infty}$ is equal to $[0,\infty)$. By Theorem 2.3, this implies that $A_f = [0,\infty)$. Similarly, considering the sequence of elements from B_f satisfying the above assumptions, we obtain $B_f = [0,\infty)$. The corollary is proved. □

Let $\mathcal{M}(\mathbb{R})$ be the set of all signed Borel measures on $\mathcal{B}(\mathbb{R})$, which are finite on compact sets. Let $\alpha \in \mathbb{R}$. Let $F_{\mu,\alpha}\colon \mathbb{R} \to \mathbb{R}$ be the distribution function corresponding to $\mu \in \mathcal{M}(\mathbb{R})$, which is defined as follows: $F_{\mu,\alpha}(x) = \mu([\alpha, x))$ if $x > \alpha$; $F_{\mu,\alpha}(x) = -\mu([x,\alpha))$ if $x < \alpha$ and $F_{\mu,\alpha}(\alpha) = 0$. Then the function $F_{\mu,\alpha}(x)$ is left continuous. Obviously for all $\alpha, \beta \in \mathbb{R}$, we have $F_{\mu,\alpha}(x) = F_{\mu,\beta}(x) + C_{\alpha,\beta}$, $x \in \mathbb{R}$, where $C_{\alpha,\beta} \in \mathbb{R}$, and $\mu([a,b)) = F_{\mu,\alpha}(b) - F_{\mu,\alpha}(a) = F_{\mu,\beta}(b) - F_{\mu,\beta}(a)$, $a, b \in \mathbb{R}, a < b$.

Let $\mu \in \mathcal{M}(\mathbb{R})$. Let F_μ be the distribution function corresponding to $\mu \in \mathcal{M}(\mathbb{R})$, which is left continuous. Then F_μ is uniquely determined up to a constant, i.e. if F_μ and $\widetilde{F_\mu}$ are two distribution functions corresponding to μ, which are left continuous, then there exists $C \in \mathbb{R}$, such that $\widetilde{F_\mu} = F_\mu + C$.

Moreover, if the function $f \in BV$ is left continuous, then we consider the signed measure $\mu \in \mathcal{M}(\mathbb{R})$ such that $\mu([a,b)) = f(b) - f(a)$, $a, b \in \mathbb{R}$, $a < b$. Then $f = F_\mu$ (up to a constant), where F_μ is the distribution function corresponding to μ, which is a left continuous function. Consequently, we can regard left continuous functions $f \in BV$ as distribution functions of signed measures $\mu \in \mathcal{M}(\mathbb{R})$.

Similarly, if $f \in BV$ is a right continuous function, then it is the distribution function of signed measure $\mu \in \mathcal{M}(\mathbb{R})$ such that $\mu((a,b]) = f(b) - f(a)$, $a, b \in \mathbb{R}$, $a < b$.

It is not difficult to prove that every function $f \in BV$ can be written in the form of the sum of left continuous and right continuous functions from BV. Thus, every function $f \in BV$ is the distribution function of the signed measure $\mu \in \mathcal{M}(\mathbb{R})$ and without loss of generality, we may assume that if $f \in BV$ then f is left continuous.

In the following theorem, we give a characterization of a-Wright convexity of functions $f \in BV$ in terms of measures μ corresponding to f such that $F_\mu = f$.

Theorem 2.6. *Let $a \geq 0$, $\mu \in \mathcal{M}(\mathbb{R})$ and $f = F_\mu$. Then f is a-Wright convex if and only if*

$$\mu(B+a) \geq \mu(B) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}). \tag{2.3}$$

Proof. If $a = 0$, then the assertion is obviously true. Assume that $a > 0$.

($\Rightarrow$) Assume that f is a-Wright convex. Let $t > 0$. Then

$$\begin{aligned} 0 \leq \Delta_t \Delta_a f(x) &= \Delta_a \Delta_t f(x) = \Delta_a (f(x+t) - f(x)) = \Delta_a \mu([x, x+t)) \\ &= \mu([x, x+t) + a) - \mu([x, x+t)). \end{aligned} \tag{2.4}$$

By (2.4), we have that inequality (2.3) is satisfied for all sets B of the form $B = [x, x+t)$, where $x \in \mathbb{R}$, $t > 0$, which implies that (2.3) is satisfied for all sets $B \in \mathcal{B}(\mathbb{R})$.

($\Leftarrow$) Assume, that (2.3) holds for all sets $B \in \mathcal{B}(\mathbb{R})$. Then in particular, it is satisfied for $B = [x, x+t)$, where $x \in \mathbb{R}, t > 0$. Then taking into account that $\mu([x, x+t) + a) < \infty$ and $\mu([x, x+t)) < \infty$, by (2.4), we obtain that f is a-Wright convex. The theorem is proved. □

We will call the measures $\mu \in \mathcal{M}(\mathbb{R})$ satisfying (2.3) *a-superinvariant measures*. We say that μ is *S-superinvariant* if it is a-superinvariant for all $a \in S$, where $S \subset [0, \infty)$.

Corollary 2.7. *Let $a \geq 0$, $\mu \in \mathcal{M}(\mathbb{R})$ and $f = F_\mu$. Then f is a-Wright concave if and only if*

$$\mu(B+a) \leq \mu(B) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}).$$

Corollary 2.8. *Let $a \geq 0$, $\mu \in \mathcal{M}(\mathbb{R})$ and $f = F_\mu$. Then*

(a) *f is a-Wright convex if and only if $\mu(B+ia) \geq \mu(B)$, $B \in \mathcal{B}(\mathbb{R})$, $i = 0, 1, 2, \ldots$,*

(b) *f is a-Wright convex if and only if $\mu(B) \geq \mu(B-ia)$, $B \in \mathcal{B}(\mathbb{R})$, $i = 0, 1, 2, \ldots$,*

(c) *f is a-Wright concave if and only if $\mu(B+ia) \leq \mu(B)$, $B \in \mathcal{B}(\mathbb{R})$, $i = 0, 1, 2, \ldots$,*

(d) *f is a-Wright concave if and only if $\mu(B) \leq \mu(B-ia)$, $B \in \mathcal{B}(\mathbb{R})$, $i = 0, 1, 2, \ldots$.*

By (2.2), we obtain immediately the following lemma.

Lemma 2.9. *Let $f \in BV$ be a function of the following form $f(x) = \int_{-\infty}^{x} g(u)\, du$, $x \in \mathbb{R}$, where $g \colon \mathbb{R} \to \mathbb{R}$ is an integrable function such that $g(x) = 0$ if $x < 0$. Let $\widetilde{f} = -f$. Then*

(a) $a \in A_f$ *if and only if* $\Delta_a g(u) \geq 0$ λ-*a.e.*,
(b) $a \in B_f$ *if and only if* $\Delta_a g(u) \leq 0$ λ-*a.e.*,
(c) $a \in A_f$ *if and only if* $a \in B_{\widetilde{f}}$.

Let $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$ $(B \subset \mathbb{R})$. We give examples of functions f and their corresponding sets A_f, B_f.

(E1) $A_f = \{0\} \cup [10, \infty)$, $B_f = \{0\}$, if $f(x) = \int_{-\infty}^{x} g(u)\, du$, where $g(x) = \chi_{[0,1]\cup[10,\infty)}(x)$ $(x \in \mathbb{R})$, as a consequence of Lemma 2.9, because $\{a \geq 0\colon \Delta_a g(u) \geq 0\ \lambda\text{-}a.e.\} = \{0\} \cup [10, \infty)$, and $\{a \geq 0\colon \Delta_a g(u) \leq 0\ \lambda\text{-}a.e.\} = \{0\}$.

By Theorem 2.6 and Corollary 2.7, we obtain

(E2) $A_f = \bigcup_{j=0}^{\infty} \{jh_0\}$, $B_f = \{0\}$, if $f = F_\mu$ and $\mu = \sum_{j=0}^{\infty} \delta_{jh_0}$, $h_0 > 0$,

(E3) $A_f = B_f = \bigcup_{j=-\infty}^{\infty} \{jh_0\}$, if $f = F_\mu$ and $\mu = \sum_{j=-\infty}^{\infty} \delta_{jh_0}$, $h_0 > 0$.

Let $\mathcal{S}$ be the set of all closed additive subsemigroups of $[0, \infty)$ containing 0. By Theorem 2.3 and Corollary 2.4, if $f \in BV$ then $A_f, B_f \in \mathcal{S}$. In the next theorem, we prove that the converse is true. Let $f \in BV$, we put $S(f) = A_f$.

Theorem 2.10. *Let $S \in \mathcal{S}$. Then there exists a function $f \in BV$ such that*

$$A_f = S, \tag{2.5}$$

$$B_{-f} = S. \tag{2.6}$$

Proof. Let $S \in \mathcal{S}$. If $S = \{0\}$, then by Theorem 2.3, for the function $f = F_\mu$ with $\mu = \delta_1$, equality (2.5) is satisfied. If $S = [0, \infty)$, then the function $f(x) = x_+ = \max(x, 0)$ $(x \in \mathbb{R})$ is of the form $f(x) = \int_{-\infty}^{x} g(u)\, du$, $x \in \mathbb{R}$, where $g(x) = \chi_{[0,\infty)}(x)$, $x \in \mathbb{R}$. Since $\{a \geq 0\colon \Delta_a g(u) \geq 0\ \lambda\text{-}a.e.\} = [0, \infty)$, by Lemma 2.9, equality (2.5) is satisfied.

Assume, that $S \neq \{0\}$ and $S \neq [0, \infty)$. First, we consider the case when the set S is of the following form

$$S = \bigcup_{r=1}^{n} A_r \cup \{0\}, \tag{2.7}$$

where $A_r = [c_r, d_r]$, $0 < c_r \leq d_r < c_{r+1} < \infty$, $r = 1, 2, \ldots, n-1$, $A_n = [c_n, \infty)$, $n \in \mathbb{N}$. Let ϵ be a real number such that $0 < \epsilon < \min_{r=0,1,\ldots,n-1}(c_{r+1} - d_r)$, where $d_0 = 0$. Given $c > 0$, we put $\omega_c(x) = \chi_{[0,c]}(x)$, $x \in \mathbb{R}$. Let $g(x) = \sup_{s \in S} \omega_\epsilon(x - s)$, $x \in \mathbb{R}$, $f(x) = \int_{-\infty}^{x} g(u) du$. Since $\{a \geq 0\colon \Delta_a g(u) \geq 0\ \lambda\text{-}a.e.\} = S$, by Lemma 2.9, equality (2.5) is satisfied.

Assume now that S is not of the form (2.7).

Assume first, that there exists $M > 0$, such that $[M, \infty) \subset S$. Then, the set $D_M = S \cap [0, M]$ is a nonempty closed set and the set $D'_M = (0, M) \setminus D_M$

is a nonempty open set. Then for every $x \in D'_M$, there exists an open interval U_x such that $x \in U_x$ and $A_x \subset D'_M$. Let $\mathcal{U}(x)$ be the set of all intervals U_x such that $x \in U_x$ and $U_x \subset D'_M$. Let $\widetilde{U_x} = \bigcup \{U_x \colon U_x \in \mathcal{U}(x)\}$, i.e. $\widetilde{U_x}$ is the biggest interval from among intervals U_x. Obviously, if $y \in \widetilde{U_x}$, then $\widetilde{U_x} = \widetilde{U_y}$ and if $y \notin \widetilde{U_x}$, then $\widetilde{U_x} \cap \widetilde{U_y} = \emptyset$. Then for all $x, y \in D'_M$, either $\widetilde{U_x} = \widetilde{U_y}$ or $\widetilde{U_x} \cap \widetilde{U_y} = \emptyset$. We have $D'_M = \bigcup \{\widetilde{U_x} \colon x \in D'_M\}$. Let $\delta > 0$. Since $D'_M \subset (0, M)$, it follows that the number of those pairwise disjoint intervals $\widetilde{U_x}$, $x \in D'_M$, for which $|\widetilde{U_x}| \geq \delta$ is finite ($|\widetilde{U_x}|$ is the lenght of the interval $\widetilde{U_x}$).

Let $Sem(B)$ ($B \in \mathcal{B}(\mathbb{R})$, $B \subset [0, \infty)$) be the smallest closed additive semigroup such that $B \cup \{0\} \subset Sem(B)$. Let $\delta > 0$. We define the set $S_{\delta,M}$ as follows

$$S_{\delta,M} = Sem\Bigg(S \setminus \bigcup_{x \in D'_M,\, |\widetilde{U_x}| \geq \delta} \widetilde{U_x}\Bigg).$$

Then $S_{\delta,M}$ is of the form (2.7), where $c_n \leq M$. Moreover, we have $S_{\delta_1,M} \supset S_{\delta_2,M}$ if $\delta_1 > \delta_2$ and $S_{\delta,M_1} \supset S_{\delta,M_2}$ if $M_1 < M_2$, which implies $S_{\delta_1,M_1} \supset S_{\delta_2,M_2}$ if $\delta_1 > \delta_2$ and $M_1 < M_2$.

Let δ_n and M_n, $n = 1, 2, \ldots$, be sequences of positive real numbers such that $\delta_n \downarrow 0$ and $M_n \uparrow \infty$. Let $S_i = S_{\delta_i, M_i}$, $i = 1, 2, \ldots$. Then $S_i \supset S_{i+1}$, $i = 1, 2, \ldots$, $S = \bigcap_{i=1}^{\infty} S_i$ and every S_i, $i = 1, 2, \ldots$, is of the form (2.7): $S_i = \bigcup_{r=1}^{n_i} A_{i,r} \cup \{0\}$, $n_i < \infty$, $A_{i,r} = [c_{i,r}, d_{i,r}]$, $0 < c_{i,r} \leq d_{i,r} < c_{i,r+1}$, $r = 1, 2, \ldots, n_i - 1$, $A_{i,n_i} = [c_{i,n_i}, \infty)$, $c_{i,n_i} \leq M_i$, $n_i \in \mathbb{N}$ and

$$\delta_i < \min_{r=0,1,\ldots,n_i-1} (c_{i,r+1} - d_{i,r}),$$

where $d_{i,0} = 0$. Let ϵ_i, $i = 1, 2, \ldots$ be the sequence of real numbers such that $\epsilon_i > \epsilon_{i+1}$, $\lim_{i\to\infty} \epsilon_i = 0$, $0 < \epsilon_i < \delta_i$.

Let $g_i(x) = \sup_{s \in S_i} \omega_{\epsilon_i}(x - s)$, $x \in \mathbb{R}$, $f_i(x) = \int_{-\infty}^{x} g_i(u)du$. Let

$$f(x) = \sum_{i=1}^{\infty} 2^{-i} f_i(x).$$

Since $\{a \geq 0 \colon \Delta_a g_i(u) \geq 0 \ \lambda\text{-a.e.}\} = S_i$, by Lemma 2.9, $S(f_i) = S_i$, $i = 1, 2, \ldots$.

Noticing, that $S_i \supset S_{i+1}$ and $\epsilon_i > \epsilon_{i+1}$, $i = 1, 2, \ldots$, we have that $S(\sum_{i=1}^{k} 2^{-i} f_i(x)) = \bigcap_{i=1}^{k} S_i = S_k$ for all $k = 1, 2, \ldots$. Taking into account that $S = \bigcap_{i=1}^{\infty} S_i$, we obtain $A_f = S(f) = S$, consequently (2.5) is satisfied. By Lemma 2.9, equality (2.6) also holds, the theorem is proved. □

Remark 2.11. Let the function $f\colon \mathbb{R} \to \mathbb{R}$ be of the form $f = \psi_1 + \psi_2$, where $\psi_1\colon \mathbb{R} \to \mathbb{R}$, $\psi_2\colon \mathbb{R} \to \mathbb{R}$ are two S-Wright convex functions such that ψ_1 is non-decreasing and ψ_2 is non-increasing. Then $f \in BV$ and f is S-Wright convex. Putting $\varphi_1 = \psi_1$ and $\varphi_2 = -\psi_2$, we obtain that f is of the form $f = \varphi_1 - \varphi_2$, where both the functions φ_1, φ_2 are non-decreasing and the functions φ_1 and $-\varphi_2$ are S-Wright convex. In the next theorem, we prove that, conversely, if $f \in BV$ and f is S-Wright convex, then there exist non-decreasing functions φ_1, φ_2 with the properties as above.

Theorem 2.12. *Let S be a set such that $S \in \mathcal{S}$ and $S \cap (0, \infty) \neq \emptyset$. Let $f \in BV$ be a S-Wright convex left continuous function and ν be the signed measure corresponding to f by the formula $\nu([a, b)) = f(b) - f(a)$, $a, b \in \mathbb{R}$, $a < b$. Then there exist*

(a) *Borel measures ν^+ and ν^- (non-negative measures), such that $\nu = \nu^+ - \nu^-$ and ν^+ and $-\nu^-$ are both S-superinvariant,*
(b) *non-decreasing functions $\varphi_1, \varphi_2\colon \mathbb{R} \to \mathbb{R}$ such that $f = \varphi_1 - \varphi_2$ and both functions φ_1 and $-\varphi_2$ are S-Wright convex.*

Proof. Let S, f and ν satisfy the assumptions of the theorem. By the Hahn decomposition theorem, there exist two sets $P, N \in \mathcal{B}(\mathbb{R})$, such that

(1) $P \cup N = \mathbb{R}$ and $P \cap N = \emptyset$.
(2) For every $B \in \mathcal{B}(\mathbb{R})$, such that $B \subset P$, one has $\nu(B) \geq 0$, i.e. P is a positive set for ν.
(3) For every $B \in \mathcal{B}(\mathbb{R})$, such that $B \subset N$, one has $\nu(B) \leq 0$, i.e. N is a negative set for ν.

Then by the Hahn-Jordan decomposition theorem, ν has a unique decomposition into difference $\nu = \nu^+ - \nu^-$ of two positive measures ν^+ and ν^- such that $\nu^+(B) = 0$ for every Borel measurable $B \subset N$ and $\nu^-(B) = 0$ for every Borel measurable $B \subset P$. These two (positive) measures ν^+ and ν^- can be defined as $\nu^+(B) = \nu(B \cap P)$ and $\nu^-(B) = -\nu(B \cap N)$. Let $\varphi_1, -\varphi_2$ be the distribution functions corresponding to ν^+ and $-\nu^-$, respectively, such that both φ_1 and $-\varphi_2$ are left-continuous. We will show that both ν^+ and $-\nu^-$ are S-superinvariant.

If the measure ν^- is the zero measure, then $\nu = \nu^+$ and $-\nu^-$ are both S-superinvariant, and both φ_1 and $-\varphi_2$ are S-Wright convex.

Similarly, if ν^+ is the zero measure, then $\nu = -\nu^-$ and ν^+ are are both S-superinvariant, and both φ_1 and $-\varphi_2$ are S-Wright convex.

Assume, that the measures ν^+ and $-\nu^-$ are both non-zero measures.

First, we will prove, that for any $B \subset P$ and $a \in S$, we have $B + a \subset P$ ν-a.e. Suppose that, on the contrary, there are an $B_0 \subset P$ and $a_0 \in S$, such that $B_0 + a_0 \not\subset P$ ν-a.e. This implies, that

$$\nu((B_0 + a_0) \cap N) \neq 0. \tag{2.8}$$

Let $B_N \subset B_0$ be the set such that

$$(B_0 + a_0) \cap N = B_N + a_0. \tag{2.9}$$

By (2.8)

$$\nu(B_N + a_0) \neq 0. \tag{2.10}$$

Since $B_N \subset B_0 \subset P$, it follows that $\nu(B_N) = \nu^+(B_N) \geq 0$. Taking into account that ν is S-superinvariant and $a_0 \in S$, we obtain

$$\nu(B_N + a_0) \geq \nu(B_N) \geq 0. \tag{2.11}$$

By (2.10) and (2.11), we have

$$\nu(B_N + a_0) > 0. \tag{2.12}$$

By (2.9), it follows that $B_N + a_0 \subset N$, which implies that $\nu(B_N + a_0) = -\nu^-(B_N + a_0) \leq 0$. Consequently, taking into account (2.10), we obtain $\nu(B_N + a_0) < 0$, which contradicts (2.12). Thus, we obtain

$$B + a \subset P, \quad \nu\text{-a.e.} \quad \text{for all } B \subset P,\ a \in S. \tag{2.13}$$

Now we will prove that ν^+ is S-superinvariant, i.e.

$$\nu^+(B + a) \geq \nu^+(B) \tag{2.14}$$

for all $B \in (B)$ and $a \in S$.

It suffices to prove (2.14) for $B \subset P$ and for $B \subset N$.

Let $B \subset P$ and $a \in S$. Then, by (2.13) $B + a \subset P$ ν-a.e., which implies

$$\nu^+(B + a) = \nu(B + a) \geq \nu(B) = \nu^+(B).$$

Let $B \subset N$ and $a \in S$. Then $\nu^+(B) = 0$. Since ν^+ is non-negative measure, it follows that $\nu^+(B + a) \geq 0$. We have, $\nu^+(B + a) \geq 0 = \nu^+(B)$.

Consequently, we obtain that ν^+ is S-superinvariant.

Similarly, we can prove that $-\nu^-$ is S-superinvariant.

Since $\varphi_1, -\varphi_2$ are the distribution functions corresponding to ν^+ and $-\nu^-$, respectively, by Theorem 2.6, the functions φ_1 and $-\varphi_2$ are both S-Wright convex. □

3. Sets A_f and B_f

In this section, we study the relationships between the sets A_f and B_f corresponding to the function $f \in BV$. Recall that two non-zero real numbers u and v are said to be *commensurable* if their ratio $\frac{u}{v}$ is a rational number; otherwise u and v are called *incommensurable.*

LEMMA 3.1. *Let $f \in BV$ be a function and ν be the signed measure corresponding to f such that $F_\nu = f$. Assume that there exist $a_1, a_2 > 0$ such that $a_1 \in A_f$ and $a_2 \in B_f$. Then one of the following conditions is satisfied*
(a) *either there exists $a_0 > 0$ such that $A_f = B_f = \{ja_0;\ j = 0, 1, \ldots\}$,*
(b) *or $A_f = B_f = [0, \infty)$.*

PROOF. Let $f \in BV$ be a function and ν be the signed measure corresponding to f such that $F_\nu = f$. Let $a_1, a_2 > 0$ be real numbers such that $a_1 \in A_f$ and $a_2 \in B_f$. Then, by Corollary 2.8,

$$\nu(B) \le \nu(B + ja_1) \le \nu(B + ja_1 - ka_2), \quad B \in \mathcal{B}(\mathbb{R}),\ j, k = 0, 1, 2, \ldots,$$

which implies

$$\nu(B) \le \nu(B + ja_1 - ka_2), \quad B \in \mathcal{B}(\mathbb{R}),\ j, k = 0, 1, 2, \ldots. \tag{3.1}$$

First, we consider the case when for all $a_1, a_2 > 0$, if $a_1 \in A_f$ and $a_2 \in B_f$, then a_1 and a_2 are commensurable.

Let $a_1, a_2 > 0$ be fixed real numbers such that $a_1 \in A_f$ and $a_2 \in B_f$. Then there exist $p, q \in \mathbb{N}$ such that $a_2 = \frac{p}{q}a_1$, where $\frac{p}{q}$ is an irreducible fraction. Then $a_2 = pa_3$, $a_1 = qa_3$, where $a_3 = \frac{a_2}{p} = \frac{a_1}{q}$. We put $a_4 = qa_2 = pa_1 = pqa_3$. Taking $j = lp$, $l = 0, 1, 2, \ldots$, $k = mq$, $m = 0, 1, 2, \ldots$, by (3.1), we obtain

$$\nu(B) \le \nu(B + (l - m)a_4), \quad B \in \mathcal{B}(\mathbb{R}),\ l, m = 0, 1, 2, \ldots,$$

which is equivalent to

$$\nu(B) \le \nu(B + ia_4), \quad B \in \mathcal{B}(\mathbb{R}),\ i = 0, \pm 1, \pm 2, \ldots, \tag{3.2}$$

as well as

$$\nu(B_1) \le \nu(B_1 - ia_4), \quad B_1 \in \mathcal{B}(\mathbb{R}),\ i = 0, \pm 1, \pm 2, \ldots. \tag{3.3}$$

Taking $B_1 = B + ia_4$, by (3.3), we obtain

$$\nu(B) \geq \nu(B + ia_4), \quad B \in \mathcal{B}(\mathbb{R}),\ i = 0, \pm 1, \pm 2, \ldots. \tag{3.4}$$

Then, by (3.2) and (3.4), we obtain

$$\nu(B) = \nu(B + ia_4), \quad B \in \mathcal{B}(\mathbb{R}),\ i = 0, \pm 1, \pm 2, \ldots.$$

Let

$$C_0 = \{a_0 > 0 \colon \nu(B) = \nu(B + ia_0),\ B \in \mathcal{B}(\mathbb{R}),\ i = 0, \pm 1, \pm 2, \ldots\}.$$

Since $a_4 \in C_0$, it follows that $C_0 \neq \emptyset$. By Corollary 2.8, we conclude that $C_0 \subset A_f$ and $C_0 \subset B_f$. Let

$$\widetilde{a_0} = \inf\{a_0 > 0 \colon a_0 \in C_0\}.$$

Obviously, $\widetilde{a_0} \in A_f \cap B_f$. We will prove that $\widetilde{a_0} > 0$. Suppose, on the contrary, that $\widetilde{a_0} = 0$. Then, there exists a sequence of positive numbers $a_n \in C_0, n = 1, 2, \ldots$, such that $\lim_{n\to+\infty} a_n = 0$. Since, $C_0 \subset A_f \cap B_f$, it follows that $a_n \in A_f \cap B_f$, $n = 1, 2, \ldots$. Taking into account, that A_f and B_f are closed additive semigrous, we conclude that $A_f = B_f = [0, \infty)$, which contradicts the assumption that every a_1 and a_2 are commensurable if $a_1, a_2 > 0$, $a_1 \in A_f$ and $a_2 \in B_f$. Thus, we obtain that $\widetilde{a_0} > 0$.

We will prove, that

$$C_0 = \{j\widetilde{a_0} \colon j = 1, 2, \ldots\}, \tag{3.5}$$

or equivalently, that if $a_0 \in C_0$, then there exists $k_0 \in \mathbb{N}$, such that

$$a_0 = k_0\widetilde{a_0}. \tag{3.6}$$

Let $a_0 \in C_0$, then

$$\nu(B) = \nu(B + ia_0), \quad B \in \mathcal{B}(\mathbb{R}),\ i = 0, \pm 1, \pm 2, \ldots. \tag{3.7}$$

Since $a_0 \in C_0 \subset B_f$ and $a_0 \in A_f$, there exist natural numbers p, q, such that $a_0 = \frac{p}{q}\,\widetilde{a_0}$ ($\frac{p}{q}$ is an irreducible fraction). If $a_0 = \widetilde{a_0}$, then (3.6) is satisfied. Assume that $a_0 \neq \widetilde{a_0}$.

If $p < q$, then $a_0 < \widetilde{a_0}$, which contradicts the definition of $\widetilde{a_0}$ as the infimum of elements from C_0. Consequently, we obtain that $p > q$.

Assume that $q > 1$. Then $1 \le [\frac{p}{q}] < \frac{p}{q}$, which implies $0 < \frac{p}{q} - [\frac{p}{q}] < 1$. By (3.7), we have

$$\nu(B) = \nu(B + a_0) = \nu(B + \frac{p}{q}\,\widetilde{a_0}), \quad B \in \mathcal{B}(\mathbb{R}). \tag{3.8}$$

Since $\widetilde{a_0} \in C_0$, it follows that $\nu(B_1) = \nu(B_1 - j\widetilde{a_0})$, $j = 0, \pm 1, \pm 2, \ldots$, $B_1 \in \mathcal{B}(\mathbb{R})$. Then taking $B_1 = B + \frac{p}{q}\,\widetilde{a_0}$ and $j = [\frac{p}{q}]$, we obtain

$$\nu\left(B + \frac{p}{q}\,\widetilde{a_0}\right) = \nu\left(B + \frac{p}{q}\,\widetilde{a_0} - \left[\frac{p}{q}\right]\widetilde{a_0}\right) = \nu\left(B + \left(\frac{p}{q} - \left[\frac{p}{q}\right]\right)\,\widetilde{a_0}\right). \tag{3.9}$$

By (3.8) and (3.9), we obtain

$$\nu\,(B) = \nu\left(B + \left(\frac{p}{q} - \left[\frac{p}{q}\right]\right)\,\widetilde{a_0}\right),$$

which implies that

$$\left(\frac{p}{q} - \left[\frac{p}{q}\right]\right)\,\widetilde{a_0} \in C_0. \tag{3.10}$$

Since $0 < \frac{p}{q} - \left[\frac{p}{q}\right] < 1$, (3.10) contradicts the definition of $\widetilde{a_0}$ as the infimum of elements from C_0. Consequently, we obtain that $q = 1$ and $a_0 = p\widetilde{a_0}$, which implies that (3.6) is satisfied with $k_0 = p$. Thus (3.5) is proved.

Now, we will prove that $A_f = C_0 \cup \{0\}$. Let $a_1 \in A_f$, $a_1 > 0$. By Corollary 2.8,

$$\nu(B) \le \nu(B + ja_1), \quad B \in \mathcal{B}(\mathbb{R}),\ j = 0, 1, 2, \ldots. \tag{3.11}$$

Suppose, that there exists $B_0 \in \mathcal{B}(\mathbb{R})$, such that

$$\nu(B_0) < \nu(B_0 + a_1). \tag{3.12}$$

Since $a_1 \in A_f$ and $\widetilde{a_0} \in B_f$, it follows that there exist natural numbers p, q such that $a_1 = \frac{p}{q}\widetilde{a_0}$. By (3.11) and (3.12), taking $j = q$, we obtain

$$\nu(B_0) < \nu(B_0 + a_1) \le \nu(B_0 + qa_1) = \nu(B_0 + q\frac{p}{q}\widetilde{a_0}) = \nu(B_0 + p\widetilde{a_0}),$$

consequently, we have

$$\nu(B_0) < \nu(B_0 + p\widetilde{a_0}). \tag{3.13}$$

Since $\widetilde{a_0} \in C_0$, we have

$$\nu(B_0) = \nu(B_0 + p\widetilde{a_0}),$$

which contradicts (3.13). Thus, we conclude, that

$$\nu(B_2) = \nu(B_2 + a_1), \quad B_2 \in \mathcal{B}(\mathbb{R}). \tag{3.14}$$

Taking in (3.14) $B_2 = B - a_1$, we obtain

$$\nu(B) = \nu(B - a_1), \quad B \in \mathcal{B}(\mathbb{R}). \tag{3.15}$$

By (3.14) and (3.15), we obtain

$$\nu(B) = \nu(B + ja_1), \quad B \in \mathcal{B}(\mathbb{R}),\ j = 0, \pm 1, \pm 2, \ldots,$$

which implies that $a_1 \in C_0$.

Similarly one can prove that $B_f = C_0 \cup \{0\}$. Thus, we obtain that in the case when for all $a_1, a_2 > 0$ such that $a_1 \in A_f$ and $a_2 \in B_f$, a_1 and a_2 are commensurable, there exists $a_0 > 0$ such that $A_f = B_f = \{ja_0;\ j = 0, 1, \ldots\}$.

Now, we consider the case when there exist $a_1, a_2 > 0$ such that $a_1 \in A_f$, $a_2 \in B_f$, and a_1, a_2 are incommensurable. By (3.1), we obtain

$$\nu(B) \leq \nu(B + a), \quad B \in \mathcal{B}(\mathbb{R}),\ a \in D, \tag{3.16}$$

where

$$D = \{ja_1 - ka_2\colon\ j, k = 0, 1, 2, \ldots\}.$$

Since a_1 and a_2 are positive and incommensurable, it follows $\overline{D} = \mathbb{R}$.

Let $t_0 \in \mathbb{R}$, then there exists a sequence $t_n \in D$ $(n = 1, 2, \ldots)$ such that $t_n \uparrow_{n\to+\infty} t_0$. Let $c, d \in \mathbb{R}$ be such that $c < d$. By (3.16), we obtain

$$\nu([c,d)) \leq \nu([c,d) + t_n) = \nu([c + t_n, d + t_n)) = f(d + t_n) - f(d + t_n). \tag{3.17}$$

Taking into account that the function f is left continuous, we have

$$\begin{aligned} \lim_{n\to+\infty} (f(d + t_n) - f(d + t_n)) &= f(d + t_0) - f(d + t_0) \\ &= \nu([c + t_0, d + t_0)) = \nu([c,d) + t_0). \end{aligned} \tag{3.18}$$

Then, by (3.17) and (3.18), we obtain

$$\nu([c,d)) \leq \nu([c,d) + t_0),$$

which implies

$$\nu(B) \le \nu(B+t), \quad B \in \mathcal{B}(\mathbb{R}),\ t \in \mathbb{R}, \tag{3.19}$$

or equivalently

$$\nu(B_2) \le \nu(B_2 - t), \quad B_2 \in \mathcal{B}(\mathbb{R}),\ t \in \mathbb{R}. \tag{3.20}$$

Taking $B_2 = B + t$, by (3.20), we obtain

$$\nu(B+t) \le \nu(B), \quad B \in \mathcal{B}(\mathbb{R}),\ t \in \mathbb{R}. \tag{3.21}$$

Then, by (3.19) and (3.21), we obtain

$$\nu(B) = \nu(B+t), \quad B \in \mathcal{B}(\mathbb{R}),\ t \in \mathbb{R}. \tag{3.22}$$

Thus, by (3.22), we conclude that $A_f = B_f = [0, \infty)$. □

As an immediate consequence of Lemma 3.1, we obtain the following theorem.

THEOREM 3.2. *Let $f \in BV$. Then one of the following conditions is fulfilled:*
(a) $A_f = B_f = \{0\}$,
(b) $A_f = \{0\}$, $B_f \cap (0, \infty) \neq \emptyset$,
(c) $A_f \cap (0, \infty) \neq \emptyset$, $B_f = \{0\}$,
(d) $A_f = B_f = \{jh_0;\ j = 0, 1, \ldots\}$, *where* $h_0 > 0$,
(e) $A_f = B_f = [0, \infty)$.

We will give a new proof of Ng's theorem on decomposition of Wright convex functions [5] as well as for the Szostok-Balcerowski theorem on monotonic differences. Let us recall the definition of the difference property. Let $\mathcal{A}$ be a class of real functions defined on $\mathbb{R}$. $\mathcal{A}$ will be said to have the *difference property*, if any function g such that for each $a \ge 0$, $\Delta_a g \in \mathcal{A}$, is of the form $g = f + A$, where $f \in \mathcal{A}$, and A is an additive function. We will need de Brujin's theorem [2], which is related to functions, which have differences from the class BV, de Brujin proved that the class BV has the difference property ([2]).

THEOREM 3.3 (de Bruijn's Theorem [2]). *Assume that $g\colon \mathbb{R} \to \mathbb{R}$ is a function such that $\Delta_a g \in BV$ for all $a > 0$. Then there exist an additive function $A\colon \mathbb{R} \to \mathbb{R}$ and a function $f\colon \mathbb{R} \to \mathbb{R}$, such that $f \in BV$ and $g = f + A$.*

Note that the original proof by Ng [5] used de Bruijn's theorem [2], which is related to functions which have continuous differences. Recently, Páles [6] gave an elementary proof of Ng's theorem.

THEOREM 3.4 (Ng's Decomposition Theorem [5]). *A function $g\colon \mathbb{R} \to \mathbb{R}$ is Wright convex if and only if there exist a convex function $f\colon \mathbb{R} \to \mathbb{R}$ and an additive function $A\colon \mathbb{R} \to \mathbb{R}$, such that $g = f + A$.*

PROOF. We give a new proof. It suffices to prove ($\Rightarrow$). Assume that $g\colon \mathbb{R} \to \mathbb{R}$ is a Wright convex function. Then, by Proposition 1.1,

$$\Delta_t \Delta_a g(x) \geq 0 \quad (t, a > 0,\ x \in \mathbb{R}). \tag{3.23}$$

By (3.23), we obtain that $A_g = [0, \infty)$ and the function $\Delta_a g(x)$ is non-decreasing for all $a > 0$, which implies that $\Delta_a g \in BV$ for all $a > 0$. Then by de Bruijn's Theorem 3.3 [2], there exist a function $f\colon \mathbb{R} \to \mathbb{R}$ and an additive function $A\colon \mathbb{R} \to \mathbb{R}$, such that $f \in BV$ and $g = f + A$. Since $\Delta_t \Delta_a A(x) = 0$ $(t, a > 0,\ x \in \mathbb{R})$, it follows that $\Delta_t \Delta_a f(x) = \Delta_t \Delta_a g(x) \geq 0$ $(t, a > 0,\ x \in \mathbb{R})$, which implies that $\Delta_a^2 f(x) \geq 0$ for all $a > 0$, consequently, f is Jensen convex. Since $f \in BV$, we have that f is locally bounded at all $x \in \mathbb{R}$. Then by theorem of Bernstein-Doetsch (cf. [3]), we obtain that f is convex. □

We give a new proof of Szostok–Balcerowski's theorem [1, Theorem 1].

THEOREM 3.5 (Szostok–Balcerowski's theorem [1, Theorem 1]). *Let $g\colon I \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(a) *for every $a > 0$ the function g_a is monotonic,*
(b) *for every $a > 0$ the function g_a is non-increasing or for every $a > 0$ the function g_a is non-decreasing.*

PROOF. Assume that (a) is satisfied, for every $a > 0$ the function g_a is monotonic, which implies that for every $a > 0$, $g_a \in BV$. Then, by de Bruijn's Theorem 3.3 [2], there exist a function $f\colon I \to \mathbb{R}$ and an additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that $f \in BV$ and $g = f + A$. Since for every $a > 0$ the function g_a is monotonic, it follows that for every $a > 0$, $\Delta_t \Delta_a g(x) \geq 0$ for all $t > 0$, $x \in I \cap (I - a - t)$, or $\Delta_t \Delta_a g(x) \leq 0$ for all $t > 0$, $x \in I \cap (I - a - t)$. Since $g = f + A$ and $\Delta_t \Delta_a A(x) = 0$ for all $t, a > 0$, $x \in \mathbb{R}$, we obtain that for every $a > 0$, $\Delta_t \Delta_a f(x) \geq 0$ for all $t > 0$, $x \in I \cap (I - a - t)$, or $\Delta_t \Delta_a f(x) \leq 0$ for all $t > 0$, $x \in I \cap (I - a - t)$. This implies that for every $a > 0$, $a \in A_f$ or $a \in B_f$, in other words, we have that $A_f \cup B_f = [0, \infty)$. By Theorem 3.2, we obtain that either $A_f = [0, \infty)$ and $B_f = \{0\}$ or $B_f = [0, \infty)$ and $A_f = \{0\}$ or $A_f = B_f = [0, \infty)$. Thus, we have that condition (b) is satisfied. The proof (b) $\Rightarrow$ (a) is obvious. □

Remark 3.6. It follows immediately from Theorem 3.5 the version of Theorem 3.5 with conditions (a') for every $a > 0$ the function g_a is strictly monotonic and (b') for every $a > 0$ the function g_a is strictly increasing or for every $a > 0$ the function g_a is strictly decreasing, in place of conditions (a), (b), answering positively the problem of T. Szostok [8, 9].

In the following theorem, we give a generalization of Theorem 3.5.

Theorem 3.7. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a function such that g is of the form $g = f + A$, where $f \in BV$ and A is an additive function. Let $S \subset [0, \infty)$ be a closed additive semigroup such that $S \cap (0, \infty) \neq \emptyset$. Then the following statements are equivalent:*

(a) *for every $a \in S$ the function g_a is monotonic,*

(b) *for every $a \in S$ the function g_a is non-increasing or for every $a \in S$ the function g_a is non-decreasing.*

Proof. Note that the function g_a is non-increasing (non-decreasing) if and only if f_a is non-increasing (non-decreasing). Therefore, it is enough to prove the theorem for $g = f \in BV$. Moreover, conditions (a) and (b) are equivalent to the following conditions (a') and (b') (respectively).

(a') $S \subset A_f \cup B_f$.

(b') $S \subset A_f$ or $S \subset B_f$.

By Theorem 3.2, two cases may occur: either (C1) $A_f \cup B_f = A_f = B_f$ or (C2) $A_f \cup B_f = A_f$ or $A_f \cup B_f = B_f$.

Let us assume that the case (C1) occurs: $A_f \cup B_f = A_f = B_f$. If (a') is satisfied, then $S \subset A_f \cup B_f = A_f = B_f$, which implies $S \subset A_f$ and $S \subset B_f$, thus (b') is satisfied. Conversely, assume that (b') is satisfied, $S \subset A_f = A_f \cup B_f$ or $S \subset B_f = A_f \cup B_f$, then obviously $S \subset A_f \cup B_f$, and (a') is satisfied.

Let us assume that the case (C2) occurs: $A_f \cup B_f = A_f$ or $A_f \cup B_f = B_f$. Assume (a'), i.e. $S \subset A_f \cup B_f$. Then if $A_f \cup B_f = A_f$, then $S \subset A_f$, and if $A_f \cup B_f = B_f$, then $S \subset B_f$, thus (b') is satisfied. Conversely, assume that (b') is satisfied: $S \subset A_f$ or $S \subset B_f$. Assume, $S \subset A_f$. Then, taking into account that $A_f \cup B_f = A_f$ or $A_f \cup B_f = B_f$, we obtain $S \subset A_f \cup B_f$, and (a') is satisfied. Similarly, if $S \subset B_f$, then (a') is satisfied. □

Remark 3.8. If $S = [0, \infty)$, then Theorem 3.5 is a special case of Theorem 3.7, but if $S = [0, \infty)$, there is no need to assume additionally that g is of the form $g = f + A$ (where $f \in BV$ and A is an additive function), because this condition on the form of g can be proved if (a) is satisfied as well if (b) is satisfied.

Remark 3.9. Is $S = [0, \infty)$ the only closed additive semigroup such that to prove that conditions (a) and (b) in Theorem 3.7 are equivalent, there is no need to assume additionally that g is of the form $g = f + A$ (where $f \in BV$ and A is an additive function)?

Acknowledgments. We would like to thank Reviewers for taking the time and effort necessary to review the manuscript. We sincerely appreciate all valuable comments and suggestions, which helped us to improve the quality of the manuscript.

References

[1] M. Balcerowski, *On a problem of T. Szostok on functions with monotonic differences*, Aequationes Math. **85** (2013), no. 1–2, 165–167.
[2] N.G. de Brujin, *Functions whose differences belong to a given class*, Nieuw Arch. Wiskunde (2) **23** (1951), 194–218.
[3] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, PWN – Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.
[4] G. Maksa and Z. Páles, *Decomposition of higher-order Wright-convex functions*, J. Math. Anal. Appl. **359** (2009), no. 2, 439–443.
[5] C.T. Ng, *Functions generating Schur-convex sums*, in: W. Walter (ed.), *General Inequalities*, 5 (Oberwolfach, 1986), International Series of Numerical Mathematics, 80, Birkhäuser, Basel-Boston, 1987, pp. 433–438.
[6] Z. Páles, *An elementary proof for the decomposition theorem of Wright convex functions*, Ann. Math. Sil **34** (2020), no. 1, 142–150.
[7] T. Rajba, *A generalization of multiple Wright-convex functions via randomization*, J. Math. Anal. Appl. **388** (2012), no. 1, 548–565.
[8] T. Szostok, *4. Problem. Report of Meeting. The Fifth Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities, February 2–5, 2005, Będlewo, Poland*, Ann. Math. Sil. **19** (2005), 65–78.
[9] T. Szostok, *5. Problem. Report of Meeting. The Forty-fourth International Symposium on Functional Equations, May 14–20, 2006, Lousiville, USA*, Aequationes Math. **73** (2007), no. 1–2, 172–200.
[10] T. Szostok, *On ω-convex functions*, in: H. Hudzik et al. (eds.), *Function Spaces IX*, Banach Center Publications, 92, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2011, pp. 351–359.
[11] E.M. Wright, *An inequality for convex functions*, Amer. Math. Monthly **61** (1954), 620–622.

Department of Mathematics
University of Bielsko-Biala
ul. Willowa 2
43-309 Bielsko-Biała
Poland
e-mail: trajba@ubb.edu.pl