

## A NOTE ON AMALGAMATED RINGS ALONG AN IDEAL

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**Abstract.** Ring properties of amalgamated products are investigated. We offer new, elementary arguments which extend results from [5] and [12] to non-commutative setting and also give new properties of amalgamated rings.

### 1. Introduction and preliminaries

All rings in this paper are associative, we do not assume they contain unity.

We write  $I \triangleleft A$ , if  $I$  is an ideal of a ring  $A$ . We say that an ideal  $I$  of  $A$  is *essential*, if  $I \cap J \neq \{0\}$  for every  $\{0\} \neq J \triangleleft A$ .

Let  $\mathcal{R}$  be a class of rings. Then:

(a)  $\mathcal{R}$  is *closed under extensions*, if the following implication holds:

$$I \triangleleft A, I \in \mathcal{R} \quad \text{and} \quad A/I \in \mathcal{R} \implies A \in \mathcal{R}.$$

(b)  $\mathcal{R}$  is *homomorphically closed*, if every homomorphic image of a ring from  $\mathcal{R}$  is in  $\mathcal{R}$ .

(c)  $\mathcal{R}$  is *closed under subrings*, if every subring of a ring belonging to  $\mathcal{R}$  is in  $\mathcal{R}$ .

(d)  $\mathcal{R}$  is *hereditary*, if  $I \triangleleft A \in \mathcal{R}$  implies that  $I \in \mathcal{R}$ .

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*Received: 05.05.2021. Accepted: 28.07.2021. Published online: 30.08.2021.*

(2020) Mathematics Subject Classification: 16S99, 16N20, 16N40, 16N60, 16P20, 16P40.

*Key words and phrases:* amalgamated ring, radical, prime ring, reduced ring, Noetherian ring.

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A class of rings  $\gamma$  is called a *radical class* or shortly a *radical* (cf. [10]), if it is closed under extensions, homomorphically closed and  $\sum\{I \triangleleft A \mid I \in \gamma\} \in \gamma$  for every ring  $A$ .

For a given ring  $A$  and a non-empty subset  $X$  of  $A$  we denote by  $l_A(X) = \{a \in A \mid aX = \{0\}\}$  the *left annihilator* of  $X$ .

We write  $R = S \oplus_{\triangleright} I$ , if  $R = S + I$ ,  $S$  is a subring of a ring  $R$ ,  $I$  is an ideal of  $R$  and  $S \cap I = \{0\}$ . Clearly  $S \oplus_{\triangleright} I/I \cong S$ .

Let  $A, B$  be associative rings,  $J$  be an ideal of  $B$  and  $f: A \rightarrow B$  a ring homomorphism (we do not assume that  $f$  preserves identity even in case of unital rings). We consider the following subring of  $A \times B$ :

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}$$

which is called the *amalgamation of  $A$  with  $B$  along  $J$  with respect to  $f$* . We will say shortly that  $A \bowtie^f J$  is the *amalgamated ring*.

The above definition was introduced by M. D'Anna, C.A. Finocchiaro and M. Fontana in case of commutative unital rings (cf. [5]). Various classical constructions can be presented as particular cases of an amalgamation, for example amalgamated duplication of a ring along an ideal (cf. [4, 7]), Nagata's idealization also called the trivial ring extension (cf. [14]),  $D+M$  constructions or CPI-extensions (cf. [5]). Moreover, the amalgamation  $A \bowtie^f J$  is related to a construction introduced by D.D. Anderson in [1] and motivated by a classical embedding of a ring without identity into a ring with identity (cf. [8]). Furthermore, there are some applications of amalgamated algebras in algebraic geometry which follow from the fact that it is possible to iterate the amalgamation of algebras and the result is still an amalgamated algebra (cf. [5]). Moreover an amalgamation can be realized as a pullback of mappings and some pullbacks give rise to amalgamated rings (cf. [5]).

There are systematic studies concerning amalgamated rings, but almost all of them are restricted to commutative rings with unity. The exception is [12], where associative unital rings are considered. For more details see for example [2, 3, 6, 9, 11, 13].

The main goal of this paper is to show that using elementary tools of a ring theory we can easily obtain generalizations of many results from other papers (see Corollary 2.7). In particular, we extend some of results from [5] and [12] to associative rings. They turn out to be direct consequences of general Theorems 2.4 and 2.5.

## 2. Main results

We start with some basic observations. The following proposition contains straightforward properties of amalgamated rings. They correspond to those presented in [5, Proposition 5.1].

**PROPOSITION 2.1.** *Let  $A, B$  be associative rings,  $J \triangleleft B$  and  $f: A \rightarrow B$  be a ring homomorphism. Then:*

- (i)  $A \cong \{(a, f(a)) \mid a \in A\}$  is a subring of  $A \bowtie^f J$ ,
- (ii)  $J \cong \{0\} \times J$  is an ideal of  $A \bowtie^f J$  and  $A \bowtie^f J / (\{0\} \times J) \cong A$ ,
- (iii)  $f^{-1}(J) \times J$  and  $f^{-1}(J) \times \{0\}$  are ideals of  $A \bowtie^f J$  and

$$A \bowtie^f J / (f^{-1}(J) \times \{0\}) \cong f(A) + J.$$

**REMARK 2.2.**

- (a) Notice that  $A \bowtie^f J \subseteq A \times (f(A) + J)$  and  $f: A \rightarrow f(A) + J \subseteq B$ . Hence, if we restrict  $B$  to  $f(A) + J$  and denote  $\bar{f}: A \rightarrow f(A) + J$ , then  $A \bowtie^f J = A \bowtie^{\bar{f}} J$ . Therefore, without loss of generality, in the definition of amalgamated rings we may write  $f(A) + J$  instead of  $B$ .
- (b) Note that  $\{(a, f(a)) \mid a \in A\} \cap (\{0\} \times J) = \{(0, 0)\}$ . Thus Proposition 2.1 (i), (ii) yields that  $A \bowtie^f J \cong A \oplus_{\triangleright} J$ .
- (c) The statement (b) implies that:
  - (i)  $J = \{0\}$  if and only if  $A \bowtie^f J \cong A$ ,
  - (ii)  $A = \{0\}$  if and only if  $A \bowtie^f J \cong J$ .

Accordingly to the above remark we will consider the situation when  $A \neq \{0\}$ ,  $J \neq \{0\}$  and  $B = f(A) + J$ . Then we say that the amalgamated ring  $A \bowtie^f J$  is *proper*.

In the following lemma we present how properties of the class  $\mathcal{R}$  affect properties of rings  $A, J$  and  $B$ . They easily follow from Proposition 2.1.

**LEMMA 2.3.** *Let  $\mathcal{R}$  be a class of rings and  $A \bowtie^f J$  a proper amalgamated ring such that  $A \bowtie^f J \in \mathcal{R}$ .*

- (i) *If  $\mathcal{R}$  is closed under subrings, then  $A \in \mathcal{R}$  and  $J \in \mathcal{R}$ .*
- (ii) *If  $\mathcal{R}$  is homomorphically closed, then  $A \in \mathcal{R}$  and  $B \in \mathcal{R}$ .*
- (iii) *If  $\mathcal{R}$  is hereditary, then  $J \in \mathcal{R}$ .*

Let the class  $\mathcal{R}$  be closed under extensions,  $A \in \mathcal{R}$  and  $J \in \mathcal{R}$ . Thus the statement (ii) of Proposition 2.1 yields that  $A \cong A \bowtie^f J / (\{0\} \times J) \in \mathcal{R}$ . Hence, by assumptions, we conclude that  $A \bowtie^f J \in \mathcal{R}$ . Therefore, by Lemma 2.3 (i), we prove the following result.

**THEOREM 2.4.** *Assume  $A \bowtie^f J$  is a proper amalgamated ring and  $\mathcal{R}$  is the class of rings closed under extensions and subrings. Then the following conditions are equivalent:*

- (i)  $A \bowtie^f J \in \mathcal{R}$ ,
- (ii)  $A \in \mathcal{R}$  and  $J \in \mathcal{R}$ .

The above theorem shows that this type of properties of  $A \bowtie^f J$  do not depend on properties of a homomorphism  $f$ .

Note that, if  $\mathcal{R}$  is closed under extensions and subrings,  $A \in \mathcal{R}$  and  $B \in \mathcal{R}$ , then  $A \times B \in \mathcal{R}$ , so  $A \bowtie^f J \in \mathcal{R}$ . However, in a view of Remark 5.5 in [5], the condition (ii) of Theorem 2.4 can not be replaced by  $A \in \mathcal{R}$  and  $B \in \mathcal{R}$ .

Below we present what additional property should have the class  $\mathcal{R}$  to get the opposite implication in Lemma 2.3 (ii).

**THEOREM 2.5.** *Assume  $A \bowtie^f J$  is a proper amalgamated ring and  $\mathcal{R}$  is the class of rings closed under extensions, homomorphically closed and hereditary. Then the following conditions are equivalent:*

- (i)  $A \bowtie^f J \in \mathcal{R}$ ,
- (ii)  $A \in \mathcal{R}$  and  $B \in \mathcal{R}$ .

**PROOF.** The implication (i)  $\implies$  (ii) follows from Lemma 2.3 (ii).

Assume that condition (ii) holds. By Proposition 2.1 we get that  $f^{-1}(J) \times J/(\{0\} \times J) \triangleleft A \bowtie^f J/(\{0\} \times J) \cong A \in \mathcal{R}$ . Since the class  $\mathcal{R}$  is hereditary,  $f^{-1}(J) \times J/(\{0\} \times J) \cong f^{-1}(J) \times \{0\} \in \mathcal{R}$ . Proposition 2.1 (iii) implies that  $A \bowtie^f J/(f^{-1}(J) \times \{0\}) \cong B \in \mathcal{R}$ . Hence, by assumption that  $\mathcal{R}$  is closed under extensions, we prove (i). □

The following conclusion is a direct consequence of Theorem 2.5.

**COROLLARY 2.6.** *Let  $\gamma$  be a hereditary, radical class and  $A \bowtie^f J$  a proper amalgamated ring. Then the following conditions are equivalent:*

- (i)  $A \bowtie^f J \in \gamma$ ,
- (ii)  $A \in \gamma$  and  $B \in \gamma$ .

The fact below presents some applications of Theorems 2.4 and 2.5.

**COROLLARY 2.7.**

(1) *If  $\mathcal{R}$  is one of the classes of rings listed below:*

- (i) *the class of reduced rings,*
- (ii) *the class of nilpotent rings,*
- (iii) *the class of nil rings.*

*Then, by Theorem 2.4, we get a description of amalgamated rings belonging to  $\mathcal{R}$ .*

(2) If  $\mathcal{R}$  is one of the classes of rings listed below:

- (i) the class of Noetherian rings,
- (ii) the class of Artinian rings,
- (iii) the class of von Neumann regular rings,
- (iv) the class of strongly regular rings,
- (v) the Jacobson radical class,
- (vi) the prime radical class,
- (vii) the generalized nil radical class,
- (viii) the Brown-McCoy radical class,
- (ix) the Levitzki radical class.

Then, by Theorem 2.5, we get a description of amalgamated rings belonging to  $\mathcal{R}$ .

Some of the classes presented above were considered in [3, 5] and [12]. Hence we obtain generalizations of Propositions: 5.6 in [5], 2.1 in [12] and 2.21 in [3]. Moreover our arguments are different and much simpler than those in [3, 5] and [12].

Let  $I := f^{-1}(J) \times \{0\}$  and  $K := \{0\} \times J$ . Then  $I \cdot K = \{(0, 0)\}$ . This equality will turn out to be crucial in a description of primness of amalgamated rings.

The following result generalizes Theorems 2.3 and 2.4 in [12] and Proposition 5.2 in [5]. Our proof is based on the above observation, so it is completely different and much shorter than those in [12] and [5].

**THEOREM 2.8.** *Assume  $A \bowtie^f J$  is a proper amalgamated ring. Then the following conditions are equivalent:*

- (i)  $A \bowtie^f J$  is a prime ring (a domain).
- (ii)  $B$  is a prime ring (a domain) and  $f^{-1}(J) = \{0\}$ .

**PROOF.** (i)  $\implies$  (ii) Assume  $A \bowtie^f J$  is a prime ring. By Proposition 2.1 (ii), (iii),  $I := f^{-1}(J) \times \{0\} \triangleleft A \bowtie^f J$  and  $K := \{0\} \times J \triangleleft A \bowtie^f J$ . Since  $I \cdot K = \{(0, 0)\}$ , so the primness of the ring  $A \bowtie^f J$  implies that  $I = \{(0, 0)\}$  or  $K = \{(0, 0)\}$ .

If  $K = \{(0, 0)\}$ , then  $J = \{0\}$ , a contradiction. Therefore  $I = \{(0, 0)\}$ , which yields that  $f^{-1}(J) = \{0\}$ . Hence  $A \bowtie^f J \cong B$ , by the statement (iii) of Proposition 2.1. Thus  $B$  is a prime ring.

If  $A \bowtie^f J$  is a domain, then clearly it is a prime ring. Hence applying the above argumentation we show that  $B$  is a domain and  $f^{-1}(J) = \{0\}$ .

The implication (ii)  $\implies$  (i) follows directly from Proposition 2.1 (iii).  $\square$

Below we collect well-known facts concerning prime rings and domains.

LEMMA 2.9. *Let  $\{0\} \neq I \triangleleft R$ .*

- (1) *The following conditions are equivalent.*
- (i)  *$R$  is a prime ring.*
  - (ii)  *$I$  is a prime ring and  $I$  is an essential ideal of  $R$ .*
  - (iii)  *$I$  is prime ring and  $l_R(I) = \{0\}$ .*
- (2) *If  $I$  is an essential ideal of  $R$  and  $I$  is a domain, then  $R$  is a domain.*

By the above lemma and Theorem 2.8 we obtain the following two results.

COROLLARY 2.10. *Assume  $A \bowtie^f J$  is a proper amalgamated ring. Then the following conditions are equivalent:*

- (i)  *$A \bowtie^f J$  is a prime ring.*
- (ii)  *$J$  is a prime ring,  $l_B(J) = \{0\}$  and  $f^{-1}(J) = \{0\}$ .*
- (iii)  *$J$  is a prime ring,  $J$  is an essential ideal of  $B$  and  $f^{-1}(J) = \{0\}$ .*

COROLLARY 2.11. *Assume  $A \bowtie^f J$  is a proper amalgamated ring. Then the following conditions are equivalent:*

- (i)  *$A \bowtie^f J$  is a domain.*
- (ii)  *$J$  is a domain,  $l_B(J) = \{0\}$  and  $f^{-1}(J) = \{0\}$ .*

PROOF. Assume (i) holds. Then  $J$  is a domain and  $A \bowtie^f J$  is a prime ring. Hence, by Corollary 2.10, we get (ii).

Suppose (ii) is true. Since  $J$  is a domain, it is also a prime ring. Thus Corollary 2.10 implies that  $J$  is an essential ideal of  $B$ . Therefore, by the statement (2) of Corollary 2.10, we conclude that  $B$  is a domain. Finally, applying Theorem 2.8 we obtain (i). □

**Acknowledgments.** The autor would like to thank Professor Jerzy Matczuk for many helpful comments and suggestions which improved the paper in a significant way.

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