Annales Mathematicae Silesianae **34** (2020), no. 1, 151–163
DOI: 10.2478/amsil-2020-0002

CONNECTIONS BETWEEN THE COMPLETION OF NORMED SPACES OVER NON-ARCHIMEDEAN FIELDS AND THE STABILITY OF THE CAUCHY EQUATION

Jens Schwaiger

Dedicated to Zygfryd Kominek with best wishes on occasion of his 75th birthday

Abstract. In [12] a close connection between stability results for the Cauchy equation and the completion of a normed space over the rationals endowed with the usual absolute value has been investigated. Here similar results are presented when the valuation of the rationals is a p-adic valuation. Moreover a result by Zygfryd Kominek ([5]) on the stability of the Pexider equation is formulated and proved in the context of Banach spaces over the field of p-adic numbers.

1. Introduction and preliminaries

Let G be an abelian semigroup and X a normed space over $\mathbb{Q}$. For $f \in X^G$ let $\gamma_f : G \times G \to X$ be defined by $\gamma_f(x,y) := f(x+y) - f(x) - f(y)$. Then we define

$$\mathscr{A}(G,X) := \{f \in X^G \mid \|\gamma_f\|_\infty < \infty\},$$

Received: 05.11.2019. Accepted: 29.02.2020. Published online: 08.05.2020.
(2010) Mathematics Subject Classification: 39B82, 46S10.
Key words and phrases: Cauchy equation, Hyers-Ulam stability, p-adic functional analysis.

where $\|\gamma_f\|_\infty := \sup\{\|\gamma_f(x,y)\| \mid x,y \in G\}$. Moreover

$$\mathscr{B}(G,X) := \{f \in X^G \mid \|f\|_\infty < \infty\}.$$

$\mathscr{A}(G,X)$ is a subspace of the rational vector space X^G containing $\mathscr{B}(G,X)$ as a subspace. [12, Sec. 12.3] contains the following result.

THEOREM 1.1. *Let G be an abelian semigroup, suppose X to be a normed vector space (over $\mathbb{Q}$) with completion X^c. Then*

$$\mathscr{A}(G,X)/\mathscr{B}(G,X) \cong Hom(G,X^c),$$

the group of homomorphisms defined on G with values in X^c.

In [11] the author investigated certain stability questions in such a way that besides the ordinary absolute value on $\mathbb{Q}$ also others, and by Ostrowski's Theorem ([9]) essentially all non-trivial valuations, have been taken into account. Each of those other valuations depend on one prime number p and are defined by

$$|0|_p := 0, \quad \left|p^\alpha \frac{a}{b}\right|_p := p^{-\alpha},$$

where a, b are integers $\neq 0$ and not divisible by p. These valuations satisfy

$$\begin{aligned} &|x|_p \geq 0, \quad |x|_p = 0 \Longleftrightarrow x = 0,\\ &|xy|_p = |x|_p\,|y|_p\,, \quad |x+y|_p \leq \max\{|x|_p\,, |y|_p\}. \end{aligned}$$

The latter property is the *ultrametric* property or *strong* triangle inequality. It is worthwhile to note that $|n|_p \leq 1$ for all integers n and $0 < |n|_p < 1 \Longleftrightarrow p \mid n, n \neq 0$. The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to $|\ |_p$ is again a field, the field of p-adic numbers.

Normed spaces and Banach spaces over $(\mathbb{Q}, |\ |_p)$ and $(\mathbb{Q}_p, |\ |_p)$ may be defined as usual. If the norm also satisfies the strong triangle inequality these spaces are called non-archimedean normed and non-archimedean Banach spaces respectively. In the literature on non-archimedean functional analysis usually only this type of norm is considered (see [8], for example).

REMARK 1.2. Let $X := \mathbb{Q}_p^{(\mathbb{N})} := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}_p^{\mathbb{N}} \mid x_n = 0$ for all but finitely many $n\}$. Then $\|\ \|_1\,, \|\ \|_2$ with $\|(x_n)_{n\in\mathbb{N}}\|_1 := \max_{n\in\mathbb{N}}\{|x_n|_p\}$ and $\|(x_n)_{n\in\mathbb{N}}\|_2 := \sum_{n\in\mathbb{N}} |x_n|_p$ are two norms. The first one is non-archimedean, the second not, and the induced topologies are different.

The first assertions may be seen immediately. The last one follows from the fact, that the sequence of the $x^{(n)} := (\underbrace{p^n, p^n, \ldots, p^n}_{p^n-\text{times}}, 0, \ldots)$ converges to 0 with respect to $\| \ \|_1$ and that $\left\|x^{(n)}\right\|_2 = 1$ for all n.

Therefore it may happen that a norm is not equivalent to a non-archimedean one. But as in the archimedean case in every finite dimensional normed space X over $\mathbb{Q}_p$ any two norms are equivalent. This implies that every norm is equivalent to a non-archimedean one. One of these may be defined by $\|\sum_{i=1}^n \xi_i e_i\| := \max_{1\le i\le n} |\xi_i|_p$ for a given basis $\{e_1, e_2, \ldots, e_n\}$ of X.

[1, TVS I.6] contains the fact, that the completion of a normed space over $(\mathbb{Q}, | \ |_p)$ is also a Banach space over $(\mathbb{Q}_p, | \ |_p)$. Moreover the completion of a non-archimedean normed space is a non-archimedean Banach space.

2. A general stability result for the Cauchy equation

Quite some years ago it became fashionable to consider stability of functional equations with a fixed bound replaced by one depending on the variables involved (and satisfying certain conditions). A very general (and therefore not widely noticed) result is to be found in [2]. A later paper ([4]) has been the base for many papers of similar results. Here is one of those.

THEOREM 2.1. *Let S be a commutative semigroup which is uniquely divisible by the prime p, i.e., the mapping $S \ni x \mapsto px =: \alpha(x) \in S$ is bijective, let X be a normed space over $(\mathbb{Q}, | \ |_p)$ with completion X^c. Assume moreover that $\varphi\colon S \times S \to [0, \infty)$ satisfies*

(i) $\lim_{n\to\infty} \dfrac{\varphi(\frac{x}{p^n}, \frac{y}{p^n})}{p^n} = 0, \ x, y \in S,$

(ii) $\Phi(x) := \sum_{n=0}^{\infty} \frac{1}{p^n} \varphi_p(\frac{x}{p^n}) < \infty, \ x \in S,$

where $\varphi_p(x) := \sum_{j=1}^{p-1} \varphi(jx, x)$ and $\frac{x}{p^n} := \alpha^{-n}(x)$. Then, given $f\colon S \to X$ such that

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y), \quad x, y \in S, \tag{2.1}$$

there is an additive function $a\colon S \to X^c$ satisfying

$$\|f(x) - a(x)\| \le \Phi(x), \quad x \in S. \tag{2.2}$$

If moreover an additive function $b\colon S \to X^c$ fulfils the inequality

$$\|f(x) - b(x)\| \le k\Phi(x)$$

for all x with $k > 0$, then $b = a$.

Proof. Putting $y = x$ in (2.1), we obtain $\|f(2x) - 2f(x)\| \le \varphi(x,x)$. Given $n \in \mathbb{N}$ we get by using (2.1) again that

$$\begin{aligned}\|f((n+1)x) - (n+1)f(x)\| &\le \|f(nx + x) - f(nx) - f(x)\| \\ &\quad + \|f(nx) - nf(x)\|,\end{aligned}$$

implying that

$$\|f(nx) - nf(x)\| \le \sum_{j=1}^{n-1} \varphi(jx, x) =: \varphi_n(x), \quad n \in \mathbb{N}, x \in S. \tag{2.3}$$

Now, let $f_n(x) := p^n f\left(\frac{x}{p^n}\right)$. Then (2.3) implies

$$\begin{aligned}\|f_n(x) - f_{n+1}(x)\| &= \left\|p^n f\left(\frac{x}{p^n}\right) - p^{n+1} f\left(\frac{x}{p^{n+1}}\right)\right\| \\ &= |p^n|_p \left\|f\left(\frac{x}{p^n}\right) - pf\left(\frac{\frac{x}{p^n}}{p}\right)\right\| \\ &\le p^{-n}\varphi_p\left(\frac{x}{p^n}\right).\end{aligned}$$

Thus

$$\|f_n(x) - f_{n+m}(x)\| \le \sum_{j=0}^{m-1} p^{-(n+j)}\varphi_p\left(\frac{x}{p^{n+j}}\right), \quad x \in S, \tag{2.4}$$

for all $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, which by (ii) shows that the sequence $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let $a\colon S \to X^c$ be defined by $a(x) := \lim_{n\to\infty} f_n(x)$. (2.1) implies

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \le \frac{\varphi(\frac{x}{p^n}, \frac{y}{p^n})}{p^n}.$$

Taking the limit for $n \to \infty$ condition (i) implies that a is additive.

(2.2) results from (2.4) with $n = 0$ and taking the limit for m to ∞.

If finally an additive function b satisfies $\|f(x)-b(x)\| \le k\Phi(x)$ for all x we get $\|a(x)-b(x)\| \le (k+1)\Phi(x)$ and with $\frac{x}{p^n}$ also

$$\|a(x)-b(x)\| = \left\|p^n\left(a\left(\frac{x}{p^n}\right)-b\left(\frac{x}{p^n}\right)\right)\right\| \le p^{-n}(k+1)\Phi\left(\frac{x}{p^n}\right).$$

Now

$$p^{-n}\Phi\left(\frac{x}{p^n}\right) = \sum_{j=0}^{\infty}\frac{1}{p^{n+j}}\varphi_p\left(\frac{x}{p^{n+j}}\right) = \sum_{j=n}^{\infty}\frac{1}{p^j}\varphi_p\left(\frac{x}{p^j}\right)$$

showing that $\lim_{n\to\infty} p^{-n}\Phi(\frac{x}{p^n}) = 0$ and finally that $a = b$. □

COROLLARY 2.2. *Let S be a commutative semigroup which is uniquely divisible by the prime p, let X be a normed space over $(\mathbb{Q}, |\ |_p)$ with completion X^c. Let $\varepsilon > 0$ and assume that $f\colon S \to X$ satisfies*

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon, \quad x, y \in S.$$

Then there is an additive function $a\colon S \to X^c$ such that

$$\|f(x)-a(x)\| \le p\varepsilon, \quad x \in S. \tag{2.5}$$

If moreover an additive function $b\colon S \to X^c$, satisfies $\|f(x)-b(x)\| \le k\varepsilon$ for all x, then $b = a$.

PROOF. Let $\varphi(x,y) := \varepsilon$. Then (i) of Theorem 2.1 is satisfied. Moreover $\phi_p(x) = (p-1)\varepsilon$ and thus

$$\Phi(x) = (p-1)\frac{1}{1-\frac{1}{p}}\varepsilon = p\varepsilon.$$

Therefore the result follows from Theorem 2.1. □

In the non-archimedean case a stronger version of Theorem 2.1 my be proved.

THEOREM 2.3. *Let S be a commutative semigroup which is uniquely divisible by the prime p, let X be a non-archimedean normed space over $(\mathbb{Q}, |\ |_p)$ with completion X^c. Assume moreover that $\varphi\colon S \times S \to [0,\infty)$ satisfies*

(i') $\lim_{n\to\infty} \dfrac{\varphi(\frac{x}{p^n}, \frac{y}{p^n})}{p^n} = 0,\ x, y \in S$

(ii') $\lim_{n\to\infty} p^{-n}\varphi'_p(\frac{x}{p^n}) = 0,\ x \in S$,
where $\varphi'_p(x) := \max_{1\le j\le p-1}\varphi(jx,x)$. *Then, given* $f\colon S \to X$ *such that*

$$\|f(x+y)-f(x)-f(y)\| \le \varphi(x,y), \quad x,y \in S,$$

there is an additive function $a\colon S \to X^c$ *fulfilling*

$$\|f(x)-a(x)\| \le \Phi'(x) := \sup_{n\in\mathbb{N}_0} p^{-n}\varphi'_p\left(\frac{x}{p^n}\right), \quad x \in S. \tag{2.6}$$

If moreover an additive function $b\colon S \to X^c$ *satisfies* $\|f(x)-b(x)\| \le k\Phi'(x)$ *for all* x *with* $k>0$, *then* $b=a$.

Proof. Since we are in the non-archimedean case the estimate for $f(px)-pf(x)$ now reads as

$$\|f(px)-pf(x)\| \le \max_{1\le j\le p-1}\varphi(jx,x) = \varphi'(x).$$

This with $f_n(x) := p^n f\left(\frac{x}{p^n}\right)$ for $n\in\mathbb{N}_0$ implies

$$\begin{aligned}\|f_n(x)-f_{n+1}(x)\| &= \left\|p^n f\left(\frac{x}{p^n}\right) - p^{n+1} f\left(\frac{x}{p^{n+1}}\right)\right\| \\ &= |p^n|_p \left\| f\left(\frac{x}{p^n}\right) - pf\left(\frac{\frac{x}{p^n}}{p}\right)\right\| \\ &\le p^{-n}\varphi'_p\left(\frac{x}{p^n}\right).\end{aligned} \tag{2.7}$$

Thus by (ii') the sequence $(f_{n+1}(x)-f_n(x))_{n\in\mathbb{N}}$ is a null sequence and therefore, since we are in the non archimedean case, a Cauchy sequence. Let $a\colon S\to X^c$, $a(x) := \lim_{n\to\infty} f_n(x)$, be the limit function. Then, as in the proof of Theorem 2.1, (i') implies that a is additive. (2.7) implies

$$\begin{aligned}\|f_n(x)-f_{n+m}(x)\| &\le \max_{0\le j\le m-1}\|f_{n+j}(x)-f_{n+j+1}(x)\| \\ &\le \max_{0\le j\le m-1} p^{-(n+j)}\varphi'_p\left(\frac{x}{p^{n+j}}\right) \\ &\le \sup_{j\ge n} p^{-j}\varphi'_p\left(\frac{x}{p^j}\right), \quad n\in\mathbb{N}_0, m\in\mathbb{N}.\end{aligned}$$

For $n = 0$ and with $m \to \infty$ we get (2.6). As for the last part we have to show that an additive function $c\colon S \to X^c$ is identically 0 provided that $\|c(x)\| \leq l\Phi'(x)$ for all x. Using this inequality for $\frac{x}{p^m}$ together with the additivity of c implies $\|c(x)\| \leq \frac{1}{p^m} l\Phi'(\frac{x}{p^m}) = l \sup_{j\geq m} \frac{1}{p^j}\varphi'_p(\frac{x}{p^j})$. And this expression tends to zero for $m \to \infty$ by (ii'). □

Corollary 2.4. *Let S be a commutative semigroup which is uniquely divisible by the prime p, let X be a non-archimedean normed space over $(\mathbb{Q}, |\ |_p)$ with completion X^c. Let $\varepsilon > 0$ and assume that $f\colon S \to X$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in S.$$

Then there is an additive function $a\colon S \to X^c$ such that

$$\|f(x) - a(x)\| \leq \varepsilon, \quad x \in S.$$

If moreover an additive function $b\colon S \to X^c$ satisfies $\|f(x) - b(x)\| \leq k\varepsilon$ for all x with $k > 0$, then $b = a$.

Proof. For $\varphi(x, y) := \varepsilon$ condition (i') is fulfilled. For this φ the function φ'_p is given by the constant ε. Accordingly $\Phi'(x) = \epsilon$ for all x. □

Remark 2.5. [6] contains a stability result with certain conditions on the bounding function φ. But they are such that $\varphi =$ const. does *not* satisfy these conditions. In [7, Theorem 3.1] a stability result for the Pexider equation is given which only for $p = 2$ covers the case of a constant bound.

3. Stability of the Pexider equation

Zygfryd Kominek ([5]) gave a very general stability result in the setting of locally convex real sequentially complete vector spaces, which reads as follows.

Theorem 3.1. *Let $(S, +)$ be a commutative semigroup and let X be a sequentially complete, linear topological Hausdorff space. Assume that V is a sequentially closed, bounded, convex and symmetric with respect to zero subset of X. For arbitrary functions $f, g, h\colon S \to X$ satisfying the condition*

$$f(x+y) - g(x) - h(y) \in V, \quad x, y \in S,$$

there exist functions $f_1, g_1, h_1\colon S \to X$ *such that*

$$f_1(x+y) - g_1(x) - h_1(y) = 0, \quad x, y \in S,$$

$$f_1(x+y) - f(x+y) \in 15V, \quad g_1(x) - g(x) \in 7V,$$

$$\text{and} \quad h_1(x) - h(x) \in 7V, \quad x, y \in S.$$

For a particular case, namely the case of normed spaces over $(\mathbb{Q}, |\,|_p)$ a similar result holds true. The more general case of topological vector spaces over $(\mathbb{Q}, |\,|_p)$ will be left to others. The following corresponds to [5, Lemma, pp. 373–374].

Lemma. *Let S be a commutative semigroup, X a normed space over $(\mathbb{Q}, |\,|_p)$ with completion X^c and $\varepsilon > 0$. Assume that $f\colon S \to X$ satisfies*

$$\left\| f(x+y) - \frac{f(2x) + f(2y)}{2} \right\| \le 2\varepsilon, \quad x, y \in S.$$

Then for $x_0 \in S$ there exist an additive function $A\colon S \to X^c$ and a constant $X \ni c := 2f(2x_0) - f(4x_0)$ such that

$$\tag{3.1} \begin{aligned} &\|f(2x) - A(2x) - c\| \le (6p+2)\varepsilon \quad \text{and} \\ &\|f(x+y) - A(x+y) - c\| \le (12p+6)\varepsilon, \quad x, y \in S. \end{aligned}$$

Proof. For $x_0 \in S$ let $a(x) := f(x + 2x_0) - f(2x_0)$. Then

$$\begin{aligned} a(x+y) &- a(x) - a(y) \\ &= f(x+x_0+y+x_0) - f(x+2x_0) - f(y+2x_0) + f(2x_0) \\ &= f(x+x_0+y+x_0) - \frac{f(2(x+x_0)) + f(2(y+x_0))}{2} \\ &\quad + \frac{f(2(x+x_0)) + f(2x_0)}{2} - f(x+x_0+x_0) \\ &\quad + \frac{f(2(y+x_0)) + f(2x_0)}{2} - f(x+x_0+x_0). \end{aligned}$$

Since the norm of the expressions in the last three lines is $\le 2\varepsilon$ we get

$$\|a(x+y) - a(x) - a(y)\| \le 6\varepsilon, \; x, y \in S.$$

By (2.5) there is some additive function $A\colon S \to X^c$ such that

$$\|a(x) - A(x)\| \leq 6p\varepsilon,\ x, y \in S. \tag{3.2}$$

Now

$$\begin{aligned} A(2x) + c - f(2x) &= A(2x) - 2a(x) + 2a(x) + 2f(2x_0) - f(4x_0) - f(2x) \\ &= 2(A(x) - a(x)) + 2\left(a(x) + f(2x_0) - \frac{f(2x) + f(4x_0)}{2}\right) \\ &\quad + 2(A(x) - a(x)) \\ &\quad + 2\left(f(x + 2x_0) - \frac{f(2x) + f(4x_0)}{2}\right), \quad x \in S. \end{aligned}$$

Since $\|2(A(x) - a(x))\| \leq |2|_p\, 6p\varepsilon \leq 6p\varepsilon$ and

$$\left\|2\left(f(x + 2x_0) - \frac{f(2x) + f(4x_0)}{2}\right)\right\| \leq |2|_p\, 2\varepsilon \leq 2\varepsilon$$

we get the first part of (3.1). The second part can be derived from the following calculations.

$$\begin{aligned} &A(x + y) + c - f(x + y) \\ &\quad = A(x) + A(y) + 2f(2x_0) - f(4x_0) - f(x + y) \\ &\qquad + \frac{f(2x) + f(2y)}{2} - \frac{f(2x) + f(2y)}{2} + a(x) + a(y) - a(x) - a(y) \\ &\quad = (A(x) - a(x)) + (A(y) - a(y)) - \left(f(x + y) - \frac{f(2x) + f(2y)}{2}\right) \\ &\qquad + \left(f(x + 2x_0) - \frac{f(2x) + f(4x_0)}{2}\right) + \left(f(y + 2x_0) - \frac{f(2y) + f(4x_0)}{2}\right) \end{aligned}$$

by considering the estimates for the term in the last two lines. □

Theorem 3.2. *Let $(S, +)$ be a commutative semigroup and let X be a normed space over $(\mathbb{Q}, |\ |_p)$ with completion X^c. Let $\varepsilon > 0$. Then, for arbitrary functions $f, g, h\colon S \to X$ satisfying the condition*

$$\|f(x + y) - g(x) - h(y)\| \leq \varepsilon, \quad x, y \in S, \tag{3.3}$$

there exist functions $f_1, g_1, h_1\colon S \to X^c$ *such that*

$$f_1(x+y) - g_1(x) - h_1(y) = 0, \quad x, y \in S, \tag{3.4}$$

$$\begin{aligned} &\|f_1(x+y) - f(x+y)\| \le (48p+3)\varepsilon \quad \textit{and} \\ &\|g_1(x) - g(x)\|, \|h_1(x) - h(x)\| \le (24p+1)\varepsilon, \quad x, y \in S. \end{aligned} \tag{3.5}$$

Proof. Observe

$$\begin{aligned} f(x+y) &- \frac{f(2x)+f(2y)}{2} \\ &= \frac{1}{2}\left(f(x+y) - g(x) - h(y)\right) + \frac{1}{2}\left(f(x+y) - g(y) - h(x)\right) \\ &\quad - \frac{1}{2}\left(f(2x) - g(x) - h(x)\right) - \frac{1}{2}\left(f(2y) - g(y) - h(y)\right). \end{aligned} \tag{3.6}$$

By (3.3)

$$\|f(2x) - g(x) - h(x)\| \le \varepsilon, \quad x \in S.$$

Applying (3.6), we get

$$\left\| f(x+y) - \frac{f(2x)+f(2y)}{2} \right\| \le 4\left|\frac{1}{2}\right|_p \varepsilon \le 8\varepsilon.$$

Applying the lemma and (3.2) we get an additive function $A\colon S \to X^c$ such that

$$\|a(x) - A(x)\| \le 24p\varepsilon \quad \text{for all } x \in S,$$

where a is defined in the proof of the above lemma. Let f_1, g_1 and h_1 be functions defined by the following formulas:

$$\begin{aligned} f_1(x) &:= A(x) + 2f(2x_0) - g(2x_0) - h(2x_0), \quad x \in S, \\ g_1(x) &:= A(x) + f(2x_0) - h(2x_0), \quad x \in S, \\ h_1(x) &:= A(x) + f(2x_0) - g(2x_0), \quad x \in S. \end{aligned}$$

Then (3.4) holds true because A is additive. Moreover

$$\begin{aligned} g_1(x) - g(x) &= A(x) + f(2x_0) - h(2x_0) - g(x) \\ &= A(x) - a(x) + a(x) + f(2x_0) - h(2x_0) - g(x) \\ &= (A(x) - a(x)) + (f(x + 2x_0) - g(x) - h(2x_0)) \end{aligned}$$

implies

$$\|g_1(x) - g(x)\| \leq 24p\varepsilon + \varepsilon = (24p + 1)\varepsilon,$$

being part of (3.5). Similarly one may find the corresponding estimate for $h_1(x) - h(x)$. Finally we observe

$$\begin{aligned} f_1(x + y) - f(x + y) &= g_1(x) + h_1(y) - f(x + y) \\ &= (g_1(x) - g(x)) + (h_1(y) - h(y)) \\ &\quad - (f(x + y) - g(x) - h(y)), \end{aligned}$$

from which we deduce that

$$\|f_1(x + y) - f(x + y)\| \leq (24p + 1)\varepsilon + (24p + 1)\varepsilon + \varepsilon = (48p + 3)\varepsilon,$$

thus finishing (3.5). □

Remark 3.3. In case that X is a non-archimedean normed space a similar result with tighter bounds holds true.

4. Stability and completeness

Let as before S be an abelian semigroup and X a normed space over $(\mathbb{Q}, |\ |_p)$. For $f \in X^S$ let $\gamma_f : S \times S \to X$ be defined by $\gamma_f(x, y) := f(x + y) - f(x) - f(y)$. Then we define

$$\mathscr{A}(S, X) := \{f \in X^S \mid \|\gamma_f\|_\infty < \infty\},$$

where $\|\gamma_f\|_\infty := \sup\{\|\gamma_f(x, y)\| \mid x, y \in S\}$. Moreover

$$\mathscr{B}(S, X) := \{f \in X^S \mid \|f\|_\infty < \infty\}.$$

Now we formulate a result similar to that in [12, Sec. 12.3] for normed spaces as above.

Theorem 4.1. *Let S be an abelian semigroup, suppose X to be a normed vector space (over $(\mathbb{Q}, |\ |_p)$) with completion X^c. Then $\mathscr{B}(S,X)$ is a subspace of the rational vector space $\mathscr{A}(S,X)$. Moreover $\mathscr{A}(S,X)/\mathscr{B}(S,X) \cong$ Hom(S,X^c), the group of homomorphisms defined on S with values in X^c, the completion of X.*

Proof. It is trivial to see that $\mathscr{A}(S,X)$ is a subspace of X^S and that $\mathscr{B}(S,X)$ is a subspace of $\mathscr{A}(S,X)$.

By Corollary 2.2 and the proof of Theorem 2.1 we may find for every $f \in \mathscr{A}(S,X)$ some, more exactly, a unique $a = a_f \in \mathrm{Hom}(S,X^c)$ such that $\|f-a\|_\infty < \infty$ and a_f is given by $a_f(x) := \lim_{n\to\infty} p^n f\left(\frac{x}{p^n}\right)$. Let $\psi\colon \mathscr{A}(S,X) \to \mathrm{Hom}(S,X^c)$ be defined by $\psi(f) := a_f$. Obviously ψ is linear. Moreover $\psi(f) = 0$ for $f \in \mathscr{B}(S,X)$ by the definition of a_f. On the other hand $\psi(f) = 0$ implies $\|f\|_\infty = \|f - \psi(f)\|_\infty < \infty$. Thus $\ker(\psi) = \mathscr{B}(S,X)$. Finally given $a \in \mathrm{Hom}(S,X^c)$ by the density of X in X^c we may find for any $x \in S$ some $y =: f(x) \in X$ such that $\|a(x) - f(x)\| \le 1$. This implies that $f(x+y)-f(x)-f(y) = (f(x+y)-a(x+y)) - (f(x)-a(x)) - (f(y)-a(y))$ is bounded, i.e., $f \in \mathscr{A}(S,X)$ (and $a_f = a$). Thus ψ is surjective. And the isomorphism theorem implies that $\mathscr{A}(S,X)/\mathscr{B}(S,X) \cong \mathrm{Hom}(S,X^c)$. □

Corollary 4.2. *$\mathscr{A}(\mathbb{Q},X)/\mathscr{B}(\mathbb{Q},X) \cong$ Hom$(\mathbb{Q},X^c) \cong X^c$.*

Proof. This follows from the fact that $\mathrm{Hom}(\mathbb{Q},X^c)$ consists of all mappings $\mathbb{Q} \ni r \to rx$, where $x \in X^c$. □

It was proved for $G = \mathbb{Z}$ in [10] and for arbitrary abelian groups G containing at least one element of infinite order in [3] that the following theorem holds true.

Theorem 4.3 (Hyers' theorem and completeness for real normed spaces). *If G is an abelian group as above and X is a real normed space such that for any $f \in \mathscr{A}(G,X)$ there is some $a \in$ Hom(G,X) such that $f-a$ is bounded, then X necessarily must be complete.*

A similar result holds true for normed spaces over $(\mathbb{Q}_p, |\ |_p)$.

Theorem 4.4 (Hyers' theorem and completeness for normed spaces over $\mathbb{Q}_p$). *If X is a normed space over $\mathbb{Q}_p$ such that for any $f \in \mathscr{A}(\mathbb{Q},X)$ there is some $a \in$ Hom$(\mathbb{Q},X)$ such that $f-a$ is bounded, then X necessarily must be complete.*

Proof. Let $x \in X^c$. For every $r \in \mathbb{Q}$ there is some $x_r =: f(r) \in X$ such that
$$\|f(r) - rx\| < 1.$$

Then $f \in \mathscr{A}(\mathbb{Q}, X)$ and therefore, by assumption, there is some $x_0 \in X$ and some $\varepsilon > 0$ such that

$$\|f(r) - rx_0\| \leq \varepsilon \quad \text{for all rational numbers } r.$$

But then $\sup\{\|r(x - x_0)\| \mid r \in \mathbb{Q}\} < \infty$ which is only possible for $x = x_0$. Thus $x \in X$ and finally $X^c \subseteq X$. □

References

[1] N. Bourbaki, *Elements of Mathematics. Topological Vector Spaces*, Chapters 1–5, Transl. from the French by H.G. Eggleston and S. Madan, Springer-Verlag, Berlin, 1987.

[2] G.L. Forti, *An existence and stability theorem for a class of functional equations*, Stochastica **4** (1980), no. 1, 23–30.

[3] G.L. Forti and J. Schwaiger, *Stability of homomorphisms and completeness*, C.R. Math. Rep. Acad. Sci. Canada **11** (1989), no. 6, 215–220.

[4] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), no. 3, 431–436.

[5] Z. Kominek, *On Hyers-Ulam stability of the Pexider equation*, Demonstratio Math. **37** (2004), no. 2, 373–376.

[6] M.S. Moslehian and Th.M. Rassias, *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Discrete Math. **1** (2007), no. 2, 325–334.

[7] A. Najati and Y.J. Cho, *Generalized Hyers-Ulam stability of the Pexiderized Cauchy functional equation in non-Archimedean spaces*, Fixed Point Theory Appl. **2011**, Art. ID 309026, 11 pp.

[8] C. Perez-Garcia and W.H. Schikhof, *Locally Convex Spaces over Non-Archimedean Valued Fields*, Vol. 119, Cambridge University Press, Cambridge, 2010.

[9] W.H. Schikhof, *Ultrametric Calculus. An Introduction to p-adic Analysis. Paperback reprint of the 1984 original*, Vol. 4, Cambridge University Press, Cambridge, 2006.

[10] J. Schwaiger, *Remark on Hyers's stability theorem*, in: R. Ger, *Report of Meeting: The Twenty-fifth International Symposium on Functional Equations*, Aequationes Math. **35** (1988), no. 1, 82–124,

[11] J. Schwaiger, *Functional equations for homogeneous polynomials arising from multilinear mappings and their stability*, Ann. Math. Sil. **8** (1994), 157–171.

[12] J. Schwaiger, *On the construction of the field of reals by means of functional equations and their stability and related topics*, in: J. Brzdęk et al. (eds.), *Developments in Functional Equations and Related Topics*, Springer, Cham, 2017, pp. 275–295.

Institute of Mathematics and Scientific Computing
University of Graz
Graz
Austria
e-mail: jens.schwaiger@uni-graz.at