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Annales Mathematicae Silesianae **32** (2018), 169–200
DOI: 10.1515/amsil-2017-0006

SOLUTIONS AND STABILITY OF GENERALIZED KANNAPPAN'S AND VAN VLECK'S FUNCTIONAL EQUATIONS

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Abstract. We study the solutions of the integral Kannappan's and Van Vleck's functional equations

$$\int_S f(xyt)d\mu(t) + \int_S f(x\sigma(y)t)d\mu(t) = 2f(x)f(y), \quad x, y \in S;$$

$$\int_S f(x\sigma(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) = 2f(x)f(y), \quad x, y \in S,$$

where S is a semigroup, σ is an involutive automorphism of S and μ is a linear combination of Dirac measures $(\delta_{z_i})_{i \in I}$, such that for all $i \in I$, z_i is in the center of S. We show that the solutions of these equations are closely related to the solutions of the d'Alembert's classic functional equation with an involutive automorphism. Furthermore, we obtain the superstability theorems for these functional equations in the general case, where σ is an involutive morphism.

1. Introduction

Throughout this paper S denotes a semigroup: a set equipped with an associative operation. We write the operation multiplicatively. A function $\chi\colon S \to \mathbb{C}$ is said to be multiplicative if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$.

Received: 17.11.2016. Accepted: 03.05.2017. Published online: 05.08.2017.
(2010) Mathematics Subject Classification: 39B82, 39B32, 39B52.
Key words and phrases: Hyers–Ulam stability, semigroup, d'Alembert's equation, Van Vleck's equation, Kannappan's equation, involution, automorphism, multiplicative function, complex measure.

Let $\sigma\colon S \to S$ denotes an involutive morphism, that is σ is an involutive automorphism: $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$, or σ is an involutive anti-automorphism: $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$.

Van Vleck [35, 36] studied the continuous solutions $f\colon \mathbb{R} \to \mathbb{R}$, $f \neq 0$ of the following functional equation

$$f(x - y + z_0) - f(x + y + z_0) = 2f(x)f(y), \quad x, y \in \mathbb{R},$$

where $z_0 > 0$ is fixed. He showed that any continuous solution with minimal period $4z_0$ has to be the sine function $f(x) = \sin(\frac{\pi}{2z_0}x) = \cos(\frac{\pi}{2z_0}(x - z_0))$, $x \in \mathbb{R}$.

Kannappan [23] proved that any solution $f\colon \mathbb{R} \to \mathbb{C}$ of the functional equation

$$f(x + y + z_0) + f(x - y + z_0) = 2f(x)f(y), \quad x, y \in \mathbb{R},$$

is periodic, if $z_0 \neq 0$. Furthermore, the periodic solution has the form $f(x) = g(x - z_0)$, where g is a periodic solution of d'Alembert functional equation

$$g(x + y) + g(x - y) = 2g(x)g(y), \quad x, y \in \mathbb{R}.$$

Stetkær [31, Exercise 9.18] found the complex-valued solutions of the functional equation

$$f(xy^{-1}z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G,$$

on group G, where z_0 is a fixed element in the center of G.

Perkins and Sahoo [27] replaced the group inversion by an involutive anti-automorphism $\sigma\colon G \to G$ and they obtained the abelian, complex-valued solutions of the functional equation

$$f(x\sigma(y)z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G. \tag{1.1}$$

Stetkær [33] extends the results of Perkins and Sahoo [27] about equation (1.1) to the more general case where G is a semigroup, the solutions are not assumed to be abelian and z_0 is a fixed element in the center of G.

Recently, Bouikhalene and Elqorachi [4] obtained the solutions of an extension of Van Vleck's functional equation

$$\chi(y)f(x\sigma(y)z_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in S,$$

on semigroup S, where χ is a multiplicative function such that $\chi(x\sigma(x)) = 1$ for all $x \in S$.

During the last ten years there has been quite a development of the theory of d'Alembert's functional equation

$$g(xy) + g(x\sigma(y)) = 2g(x)g(y), \quad x, y \in G, \tag{1.2}$$

on non abelian groups. We know that the non-zero solutions of such equation for general groups, even monoids, are the normalized traces of certain representations of the group G on $\mathbb{C}^2$ [10, 11].

Stetkær [34] expressed the complex-valued solutions of Kannappan's functional equation

$$f(xyz_0) + f(x\sigma(y)z_0) = 2f(x)f(y), \quad x, y \in S, \tag{1.3}$$

on semigroups in terms of solutions of d'Alembert's functional equation (1.2).

Elqorachi [13] extended the results of Stetkær [34, 33] to the generalizations of Kannappan's functional equation

$$\int_S f(xyt)d\mu(t) + \int_S f(x\sigma(y)t)d\mu(t) = 2f(x)f(y), \quad x, y \in S, \tag{1.4}$$

and Van Vleck's functional equation

$$\int_S f(x\sigma(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) = 2f(x)f(y), \quad x, y \in S, \tag{1.5}$$

where μ is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, with z_i in the center of the semigroup S, for all $i \in I$ and where σ is an involutive anti-automorphism of S.

Related studies of functional equations like (1.4) can be found in [1, 15, 16, 17].

Studies of the stability of functional equations highlighted a phenomenon which is usually called superstability: consider the functional equation $E(f) = 0$ and assume we are in a framework where the notion of boundedness of f and of $E(f)$ makes sense. We say that the equation $E(f) = 0$ is superstable if the boundedness of $E(f)$ implies that either f is bounded or f is a solution of $E(f) = 0$. This property was first observed in [3] where Baker, Lawrence, and Zorzitto proved the following: Let V ba a vector space. If a function $f\colon V \to \mathbb{R}$ satisfies the inequality $|f(x + y) - f(x)f(y)| \leq \varepsilon$ for some $\varepsilon > 0$ and for all $x, y \in V$, then either f is bounded on V or $f(x+y) = f(x)f(y)$ for all $x, y \in V$.

The result was generalized by Baker [2], by replacing V by a semigroup and $\mathbb{R}$ by a normed algebra E, in which the norm is multiplicative, by Ger and Šemrl [20], where E is an arbitrary commutative complex semisimple

Banach algebra and by Lawrence [26] in the case where E is the algebra of all $n \times n$ matrices. Different generalizations of the result of Baker, Lawrence and Zorzitto have been obtained. We mention for example [5], [14], [19], [21], [22], [24], [25] and [28].

The first purpose of this paper is to extend the results of Stetkær [33, 34] on the Kannappan's functional equation (1.4) and Van Vleck's functional equation (1.5) to the case, where σ is an involutive automorphism of S.

By using similar methods and computations to those in [13] we prove that the solutions of (1.4) and (1.5) are also closely related to the solutions of the d'Alembert's classic functional equation (1.2) (with σ an involutive automorphism) which has not been studied much on non-abelian semigroups. Exceptions are Stetkær [30, Example 6] (continuous solutions), Sinopoulos [29] (general solutions) for a special involutive automorphism σ of the Heisenberg group. We show that any solution of (1.4) is proportional to a solution of (1.2). We prove that all solutions of the integral Van Vleck's functional equation (1.5) are abelian and as an application we obtain some results about abelian solutions of (1.2).

We do not need the crucial proposition [31, Proposition 8.14] used in the proofs of the main results in [13] and [33, 34].

The second purpose of this paper is to prove the superstability of equations (1.4) and (1.5). We show that the superstability of these functional equations is closely related to the superstability of the Wilson's classic functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in S,$$

and consequently, we obtain the superstability theorems of equations (1.4) and (1.5) on semigroups that are not necessarily abelian and where σ is an involutive morphism.

2. Integral Kannappan's functional equation on semigroups

In this section we study the complex-valued solutions of the functional equation (1.4), where σ is an involutive automorphism and μ is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i \in I$.

Throughout this paper we use in (all) proofs without explicit mentioning the assumption that for all $i \in I$, z_i is in the center of S and its consequence $\sigma(z_i)$ is in the center of S. The following lemma has been obtained in [13] for an involutive anti-automorphism σ. It is still true, when σ is an involutive automorphism. In the proof we adapt similar computations as used in [13].

LEMMA 2.1. *If* $f\colon S\to\mathbb{C}$ *is a solution of* (1.4), *then for all* $x\in S$

$$f(x)=f(\sigma(x)),$$

$$\int_S f(t)d\mu(t)\neq 0 \Longleftrightarrow f\neq 0,$$

$$\int_S\int_S f(x\sigma(t)s)d\mu(t)d\mu(s)=f(x)\int_S f(t)d\mu(t), \tag{2.1}$$

$$\int_S\int_S f(xts)d\mu(t)d\mu(s)=f(x)\int_S f(t)d\mu(t). \tag{2.2}$$

The following notations will be used later:

– $\mathcal{A}$ consists of the solutions $g\colon S\to\mathbb{C}$ of d'Alembert's functional equation (1.2) with $\int_S g(t)d\mu(t)\neq 0$ and satisfying the condition

$$\int_S g(xt)d\mu(t)=g(x)\int_S g(t)d\mu(t)\quad \text{for all } x\in S; \tag{2.3}$$

– to any $g\in\mathcal{A}$ we associate the function $Tg=\int_S g(t)d\mu(t)g : S\to\mathbb{C}$;
– $\mathcal{K}$ consists of the non-zero solutions $f\colon S\to\mathbb{C}$ of the integral Kannappan's functional equation (1.4).

In the following theorem the complex solutions of equation (1.4) are expressed by means of solutions of d'Alembert's functional equation (1.2).

THEOREM 2.2. (1) *T is a bijection of $\mathcal{A}$ onto $\mathcal{K}$. The inverse $T^{-1}\colon\mathcal{K}\to\mathcal{A}$ is given by the formula*

$$(T^{-1}f)(x)=\frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)}$$

for all $f\in\mathcal{K}$ and $x\in S$.

(2) *Any non-zero solution $f\colon S\to\mathbb{C}$ of the integral Kannappan's functional equation* (1.4) *is of the form $f=\int_S g(t)d\mu(t)g$, where $g\in\mathcal{A}$. Furthermore, $f(x)=\int_S g(xt)d\mu(t)=\int_S g(x\sigma(t))d\mu(t)=\int_S g(t)d\mu(t)g(x)$ for all $x\in S$.*

(3) *f is central, i.e. $f(xy)=f(yx)$ for all $x,y\in S$, if and only if g is central.*

(4) *If S is equipped with a topology and $\sigma\colon S\to S$ is continuous then f is continuous if and only if g is continuous.*

Proof. Similar computations to those of [13], where σ anti-automorphism involutive, can be adapted to the present situation. The only assertion we need to prove is that the function

$$g(x) = \frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)}$$

defined in [13] satisfies the condition (2.3).

By replacing x by xks and y by r in (1.4) and integrating the result with respect to k, s and r we get

$$\begin{aligned}(2.4)\quad &\int_S\int_S\int_S\int_S f(xksrt)d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\ &\quad+\int_S\int_S\int_S\int_S f(xks\sigma(r)t)d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\ &\quad= 2\int_S\int_S f(xks)d\mu(k)d\mu(s)\int_S f(r)d\mu(r)\\ &\quad= 2f(x)\Big(\int_S f(s)d\mu(s)\Big)^2.\end{aligned}$$

By replacing x by xs and y by kr in (1.4) and integrating the result with respect to k, s and r we obtain

$$\begin{aligned}(2.5)\quad &\int_S\int_S\int_S\int_S f(xskrt)d\mu(s)d\mu(k)d\mu(r)d\mu(t)\\ &\quad+\int_S\int_S\int_S\int_S f(xs\sigma(k)\sigma(r)t)d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\ &\quad=\int_S\int_S\int_S\int_S f(xskrt)d\mu(s)d\mu(k)d\mu(r)d\mu(t)\\ &\quad+\int_S\int_S\int_S\int_S f(xs\sigma(r)\sigma(k)t)d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\ &\quad= 2\int_S\int_S f(kr)d\mu(k)d\mu(r)\int_S f(xs)d\mu(s).\end{aligned}$$

From (2.1) and (2.2) we have

$$\int_S\int_S\int_S\int_S f(xks\sigma(r)t)d\mu(k)d\mu(s)d\mu(r)d\mu(t)$$
$$= \int_S\int_S f(xks)d\mu(k)d\mu(s)\int_S f(s)d\mu(s) = f(x)\Big(\int_S f(s)d\mu(s)\Big)^2$$

and

$$\int_S\int_S\int_S\int_S f(xr\sigma(k)t\sigma(s))d\mu(k)d\mu(s)d\mu(r)d\mu(t)$$
$$= \int_S\int_S\int_S\int_S f(xr\sigma(k))d\mu(r)d\mu(k)\int_S f(s)d\mu(s) = f(x)\Big(\int_S f(s)d\mu(s)\Big)^2.$$

In view of (2.4) and (2.5) we deduce that

$$\int_S\int_S f(kr)d\mu(k)d\mu(r)\int_S f(xs)d\mu(s) = \int_S\int_S f(xks)d\mu(k)d\mu(s)\int_S f(r)d\mu(r).$$

So, by using the expression of g we obtain

$$\int_S g(xs)d\mu(s) = g(x)\int_S f(s)d\mu(s)$$

for all $x \in S$. This completes the proof. □

Remark 2.3. In Stetkær's paper [34] about Kannappan's functional equation on semigroups, in the definition of the set $\mathcal{A}$ other assertions – equivalent to (2.3) – are needed to prove the main result in [34]. We notice here that we do not need these statements. The same is also valid for the manuscript [13].

Now, we extend Stetkær's result [34] from anti-automorphisms to the more general case of morphism, as follows.

Corollary 2.4. *Let z_0 be a fixed element in the center of a semigroup S and let σ be an involutive morphism of S. Then, any non-zero solution $f\colon S \to \mathbb{C}$ of the functional equation* (1.3) *is of the form $f = g(z_0)g$, where g is a solution of d'Alembert's functional equation* (1.2) *with $g(z_0) \neq 0$ and satisfying the condition $g(xz_0) = g(z_0)g(x)$ for all $x \in S$.*

COROLLARY 2.5. *If $\sigma = I$, where I is the identity map of S, then, any non-zero solution $f\colon S \to \mathbb{C}$ of Kannappan's functional equation*

$$\int_S f(xyt)d\mu(t) = f(x)f(y), \quad x, y \in S,$$

is of the form $f = \chi \int_S \chi(t)d\mu(t)$, where χ is a multiplicative function such that $\int_S \chi(t)d\mu(t) \neq 0$.

REMARK 2.6. The result stated in Corollary 2.5 is also true without the assumption that μ is a linear combination of Dirac measures δ_{z_i} with z_i in the center of S (see [18]).

COROLLARY 2.7. *The non-zero central solutions of the integral Kannappan's functional equation* (1.4)*, where σ is an involutive automorphism of S, are the functions of the form*

$$f(x) = \left[\frac{\chi(x) + \chi(\sigma(x))}{2}\right] \int_S \chi(t)d\mu(t), \quad x \in S,$$

where $\chi\colon S \to \mathbb{C}$ is a multiplicative function such that

$$\int_S \chi(t)d\mu(t) \neq 0 \quad \textit{and} \quad \int_S \chi(\sigma(t))d\mu(t) = \int_S \chi(t)d\mu(t).$$

PROOF. From Theorem 2.2, if f is a central solution of (1.4) then g is a central solution of d'Alembert's functional equation (1.4), with σ an involutive automorphism of S. In view of [32], there exists a non-zero multiplicative function $\chi\colon S \to \mathbb{C}$ such that

$$g(x) = \frac{\chi(x) + \chi(\sigma(x))}{2} \tag{2.6}$$

for all $x \in S$. So, $f(x) = [\frac{\chi(x)+\chi(\sigma(x))}{2}] \int_S \chi(t)d\mu(t)$ with $\int_S \chi(t)d\mu(t) \neq 0$. On the other hand by substituting the condition $\int_S g(xt)d\mu(t) = g(x)\int_S g(t)d\mu(t)$ into (2.6) we get $\int_S \chi(\sigma(t))d\mu(t) = \int_S \chi(t)d\mu(t)$. This completes the proof. $\square$

3. Superstability of the integral Kannappan's functional equation (1.4)

In this section we obtain the superstability result of equation (1.4) on semigroups, not necessarily abelian. Later, we will need the following lemma.

LEMMA 3.1. *Let σ be an involutive morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i \in I$. Let $\delta > 0$ be fixed. If $f : S \to \mathbb{C}$ is an unbounded function which satisfies the inequality*

$$\left|\int_S f(xyt)d\mu(t) + \int_S f(x\sigma(y)t)d\mu(t) - 2f(x)f(y)\right| \leq \delta \tag{3.1}$$

for all $x, y \in S$, then, for all $x \in S$

$$f(\sigma(x)) = f(x), \tag{3.2}$$

$$\left|\int_S\int_S f(x\sigma(s)t)d\mu(s)d\mu(t) - f(x)\int_S f(t)d\mu(t)\right| \leq \frac{\delta}{2}\|\mu\|, \tag{3.3}$$

$$\left|\int_S\int_S f(xst)d\mu(s)d\mu(t) - f(x)\int_S f(t)d\mu(t)\right| \leq \frac{3\delta}{2}\|\mu\|, \tag{3.4}$$

$$\int_S f(t)d\mu(t) \neq 0. \tag{3.5}$$

The function g defined by

$$g(x) = \frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)} \quad \textit{for } x \in S \tag{3.6}$$

is unbounded on S and satisfies the following inequalities:

$$|g(xy) + g(x\sigma(y)) - 2g(x)g(y)| \leq \frac{3\delta}{(|\int_S f(s)d\mu(s)|)^2}\|\mu\|^2, \tag{3.7}$$

$$\begin{aligned}&\left|\int_S g(xt)d\mu(t) - g(x)\int_S g(t)d\mu(t)\right| \\ &\qquad \leq \frac{(5/4)\delta\|\mu\|^3 + (1/4)\delta\|\mu\|^2}{(|\int_S f(s)d\mu(s)|)^2} + \frac{\delta\|\mu\|}{|\int_S f(s)d\mu(s)|}\end{aligned} \tag{3.8}$$

for all $x, y \in S$. Furthermore, g is a non-zero solution of d'Alembert's functional equation (1.2) *and satisfies the condition* (2.3). *That is $T^{-1}f = g \in \mathcal{A}$.*

Proof. Equation (3.2): Replacing y by $\sigma(y)$ in (3.1) and subtracting resulting inequalities we find, after using the triangle inequality, that $|f(x)(f(y) - f(\sigma(y)))| \leq 2\delta$. Since f is assumed to be unbounded then $f(\sigma(y)) = f(y)$ for all $y \in S$.

Inequality (3.3): Replacing x by $\sigma(s)$ in (3.1) and integrating the result with respect to s we get

$$\Big|\int_S\int_S f(\sigma(s)yt)d\mu(t)d\mu(s) + \int_S\int_S f(\sigma(s)\sigma(y)t)d\mu(t)d\mu(s) - 2f(y)\int_S f(\sigma(s))d\mu(s)\Big| \leq \delta\|\mu\|,$$

which can be written

$$\Big|\int_S\int_S f(\sigma(s)yt)d\mu(t)d\mu(s) + \int_S\int_S f(\sigma(s)yt)d\mu(t)d\mu(s) - 2f(y)\int_S f(s)d\mu(s)\Big| \leq \delta\|\mu\|,$$

because $f \circ \sigma = f$. This proves (3.3).

Inequality (3.4): By setting $y = s$ in (3.1) and integrating the result with respect to s we get

$$\Big|\int_S\int_S f(xst)d\mu(t)d\mu(s) + \int_S\int_S f(x\sigma(s)t)d\mu(t)d\mu(s) - 2f(x)\int_S f(s)d\mu(s)\Big| \leq \delta\|\mu\|.$$

According to (3.3) and the triangle inequality we deduce (3.4).

Condition (3.5): Assume that f is an unbounded function which satisfies the inequality (3.1) and that $\int_S f(t)d\mu(t) = 0$. Replacing x by xs, y by yk in (3.1) and integrating the result with respect to s and k we get

$$\begin{aligned}(3.9)\quad \Big|\int_S\int_S\int_S & f(xsykt)d\mu(t)d\mu(s)d\mu(k) \\ &+ \int_S\int_S\int_S f(xs\sigma(yk)t)d\mu(t)d\mu(s)d\mu(k) \\ &- 2\int_S f(xs)d\mu(s)\int_S f(yt)d\mu(t)\Big| \leq \delta\|\mu\|^2.\end{aligned}$$

In view of (3.3) and (3.4) we have

$$\Big|\int_S\int_S\int_S f(xs\sigma(t)\sigma(y)k)d\mu(t)d\mu(s)d\mu(k) \\ -\int_S\Big[\int_S f(t)d\mu(t)f(xs\sigma(y))\Big]d\mu(s)\Big| \le \frac{\delta}{2}\|\mu\|^2,$$

$$\Big|\int_S\int_S\int_S f(xsytk)d\mu(t)d\mu(s)d\mu(k) \\ -\int_S f(t)d\mu(t)\int_S f(xys)d\mu(s)\Big| \le \frac{3\delta}{2}\|\mu\|^2.$$

Since $\int_S f(t)d\mu(t)=0$, then we get

$$\Big|\int_S\int_S\int_S f(xs\sigma(t)\sigma(y)k)d\mu(t)d\mu(s)d\mu(k)\Big| \le \frac{\delta}{2}\|\mu\|^2,$$
$$\Big|\int_S\int_S\int_S f(xsytk)d\mu(t)d\mu(s)d\mu(k)\Big| \le \frac{3\delta}{2}\|\mu\|^2.$$

From (3.9) we conclude that the function $h(x)=\int_S f(xs)d\mu(s)$ is a bounded function on S, in particular the functions $(x,y)\to\int_S f(xys)d\mu(s)$; $(x,y)\to\int_S f(x\sigma(y)s)d\mu(s)$ are bounded on $S\times S$. So, from (3.1) we deduce that f is bounded, which contradicts the assumption that f is an unbounded function on S and this proves (3.5).

Inequality (3.7): In the following we will show that the function g defined by (3.6) is unbounded. If g is bounded, then there exists $M>0$ such that $|\int_S f(xs)d\mu(s)|\le M$ for all $x\in S$. From (3.1) and the triangle inequality we get that the function $(x,y)\to f(x)f(y)$ is bounded on $S\times S$ and this implies that f is bounded. This contradicts the fact that f is assumed to be unbounded on S.

From the inequalities (3.1), (3.3) and (3.4), we get

$$\Big|\Big(\int_S f(s)d\mu(s)\Big)^2[g(xy)+g(x\sigma(y))-2g(x)g(y)]\Big| \\ =\Big|\int_S f(s)d\mu(s)\int_S f(xyt)d\mu(t)+\int_S f(s)d\mu(s)\int_S f(x\sigma(y)t)d\mu(t) \\ -2\int_S f(xk)d\mu(k)\int_S f(ys)d\mu(s)\Big|$$

$$= \Big| \int_S f(s)d\mu(s) \int_S f(xyt)d\mu(t) - \int_S \int_S \int_S f(xytks)d\mu(t)d\mu(s)d\mu(k)$$

$$+ \int_S f(s)d\mu(s) \int_S f(x\sigma(y)t)d\mu(t) - \int_S \int_S \int_S f(x\sigma(y)t\sigma(s)k)d\mu(t)d\mu(s)d\mu(k)$$

$$+ \int_S \int_S \int_S f(xk\sigma(ys)t)d\mu(t)d\mu(s)d\mu(k) + \int_S \int_S \int_S f(xkyst)d\mu(t)d\mu(s)d\mu(k)$$

$$- 2 \int_S f(xk)d\mu(k) \int_S f(ys)d\mu(s)\Big| \leq \frac{3\delta}{2}\|\mu\|^2 + \frac{\delta}{2}\|\mu\|^2 + \delta\|\mu\|^2,$$

which gives (3.7).

Inequality (3.8): For all $x \in S$, we have

$$\int_S g(xs)d\mu(s) - g(x) \int_S g(t)d\mu(t)$$

$$= \frac{\int_S \int_S f(xst)d\mu(s)d\mu(t)}{\int_S f(s)d\mu(s)} - \frac{\int_S \int_S f(ks)d\mu(k)d\mu(s) \int_S f(xs)d\mu(s)}{(\int_S f(s)d\mu(s))^2}$$

$$= \frac{\int_S\int_S f(xst)d\mu(s)d\mu(t)\int_S f(s)d\mu(s) - \int_S\int_S f(ks)d\mu(k)d\mu(s)\int_S f(xs)d\mu(s)}{(\int_S f(s)d\mu(s))^2}.$$

Replacing x by xsk and y by r in (3.1) and integrating the result with respect to s, k and r we get

$$\Big| \int_S \int_S \int_S \int_S f(xskrt)d\mu(s)d\mu(k)d\mu(r)d\mu(t)$$

$$+ \int_S \int_S \int_S \int_S f(xsk\sigma(r)t)d\mu(s)d\mu(k)d\mu(r)d\mu(t)$$

$$- 2 \int_S \int_S f(xsk)d\mu(s)d\mu(k) \int_S f(r)d\mu(r)\Big| \leq \delta\|\mu\|^3.$$

By replacing x by xs and y by kr in (3.1) and integrating the result with respect to s, k and r we get

$$\Big| \int_S \int_S \int_S \int_S f(xskrt)d\mu(s)d\mu(k)d\mu(r)d\mu(t)$$

$$+ \int_S \int_S \int_S \int_S f(xs\sigma(k)\sigma(r)t)d\mu(s)d\mu(k)d\mu(r)d\mu(t)$$

$$- 2 \int_S \int_S f(kr)d\mu(k)d\mu(r) \int_S f(xs)d\mu(s)| \leq \delta\|\mu\|^3.$$

Note that

$$
\begin{aligned}
&2\int_S\int_S f(ks)d\mu(k)d\mu(s)\int_S f(xs)d\mu(s)-2\int_S f(xst)d\mu(s)d\mu(t)\int_S f(s)d\mu(s)\\
&=\Big[2\int_S\int_S f(ks)d\mu(k)d\mu(s)\int_S f(xs)d\mu(s)\\
&-\int_S\int_S\int_S\int_S f(xsrkt)d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\
&-\int_S\int_S\int_S\int_S f(xsr\sigma(k)\sigma(t))d\mu(k)d\mu(s)d\mu(r)d\mu(t)\Big]\\
&-\Big[2\int_S f(xst)d\mu(s)d\mu(t)\int_S f(s)d\mu(s)\\
&-\int_S\int_S\int_S\int_S f(xsrkt)d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\
&-\int_S\int_S\int_S\int_S f(xsrk\sigma(t))d\mu(k)d\mu(s)d\mu(r)d\mu(t)\Big]\\
&+\int_S\int_S\int_S\int_S f(xsr\sigma(k)\sigma(t))d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\
&-\int_S\int_S f(x\sigma(t)s)d\mu(s)d\mu(t)\int_S f(t)d\mu(t)\\
&+\int_S\int_S f(x\sigma(t)s)d\mu(s)d\mu(t)\int_S f(t)d\mu(t)-f(x)(\int_S f(t)d\mu(t))^2\\
&-\Big[\int_S\int_S\int_S f(xsrk\sigma(t))d\mu(k)d\mu(s)d\mu(r)d\mu(t)\\
&-\int_S f(xst)d\mu(s)d\mu(t)\int_S f(s)d\mu(s)\Big]\\
&-\Big[\int_S f(xst)d\mu(s)d\mu(t)\int_S f(s)d\mu(s)-f(x)(\int_S f(t)d\mu(t))^2\Big].
\end{aligned}
$$

From inequalities (3.1), (3.3), (3.4) and the above relations we get

$$
\begin{aligned}
&\Big|2\int_S\int_S f(ks)d\mu(k)d\mu(s)\int_S f(xs)d\mu(s)-2\int_S f(xst)d\mu(s)d\mu(t)\int_S f(s)d\mu(s)\Big|\\
&\leq 2\delta\|\mu\|^3+\frac{\delta}{2}\|\mu\|^3+\frac{\delta}{2}\|\mu\|\Big|\int_S f(s)d\mu(s)\Big|+\frac{\delta}{2}\|\mu\|^2+\frac{3\delta}{2}\|\mu\|\Big|\int_S f(s)d\mu(s)\Big|,
\end{aligned}
$$

which implies that

$$\Big|\int_S g(xt)d\mu(t)-g(x)\int_S g(t)d\mu(t)\Big|$$
$$\leq \frac{(5/4)\delta\|\mu\|^3+(1/4)\delta\|\mu\|^2}{(|\int_S f(s)d\mu(s)|)^2}+\frac{\delta\|\mu\|}{|\int_S f(s)d\mu(s)|}$$

and this proves (3.8). Now, since g is unbounded and satisfies the inequality (3.7), we deduce (from [6]) that g satisfies the d'Alembert's functional equation (1.2). We will show that $\int_S g(xt)d\mu(t)=g(x)\int_S g(t)d\mu(t)$ for all $x\in S$.

$$2|g(y)|\Big|\int_S g(xt)d\mu(t)-g(x)\int_S g(t)d\mu(t)\Big|$$
$$=\Big|\int_S 2g(y)g(xt)d\mu(t)-2g(x)g(y)\int_S g(t)d\mu(t)\Big|$$
$$=\Big|\int_S [g(xyt)+g(x\sigma(y)t)]d\mu(t)-\int_S g(t)d\mu(t)[g(xy)+g(x\sigma(y))]\Big|$$
$$=\Big|\int_S g(xyt)d\mu(t)-\int_S g(t)d\mu(t)g(xy)+\int_S g(x\sigma(y)t)d\mu(t)-\int_S g(t)d\mu(t)g(x\sigma(y))\Big|$$
$$\leq\Big|\int_S g(xyt)d\mu(t)-g(xy)\int_S g(t)d\mu(t)\Big|$$
$$+\Big|\int_S g(x\sigma(y)t)d\mu(t)-g(x\sigma(y))\int_S g(t)d\mu(t)\Big|.$$

In view of inequality (3.8) we obtain

$$2|g(y)|\Big|\int_S g(xt)d\mu(t)-g(x)\int_S g(t)d\mu(t)\Big|$$
$$\leq 2\Big[\frac{(5/4)\delta\|\mu\|^3+(1/4)\delta\|\mu\|^2}{(|\int_S f(s)d\mu(s)|)^2}+\frac{\delta\|\mu\|}{|\int_S f(s)d\mu(s)|}\Big].$$

Since g is an unbounded function on S then we get

$$\int_S g(xt)d\mu(t)=g(x)\int_S g(t)d\mu(t)$$

for all $x\in S$. This completes the proof. $\square$

Now, we are ready to prove the main result of the present section. We notice here that the same result has been obtained in [6] with different assumptions on μ.

THEOREM 3.2. *Let σ be an involutive morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i \in I$. Let $\delta > 0$ be fixed. If $f : S \to \mathbb{C}$ satisfies the inequality*

$$\Big|\int_S f(xyt)d\mu(t) + \int_S f(x\sigma(y)t)d\mu(t) - 2f(x)f(y)\Big| \leq \delta \tag{3.10}$$

for all $x, y \in S$, then either f is bounded and $|f(x)| \leq \frac{\|\mu\|+\sqrt{\|\mu\|^2+2\delta}}{2}$ for all $x \in S$ or f is a solution of the integral Kannappan's functional equation (1.4).

PROOF. Assume that f is an unbounded solution of (3.10). Replacing y by ys in (3.10) and integrating the result with respect to s we get

$$\Big|\int_S\int_S f(xyst)d\mu(s)d\mu(t) + \int_S\int_S f(x\sigma(y)\sigma(s)t)d\mu(s)d\mu(t) - 2f(x)\int_S f(ys)d\mu(s)\Big| \leq \delta\|\mu\|$$

for all $y \in S$. From (3.3), (3.4) and the triangle inequality we get

$$\Big|\int_S f(s)d\mu(s)f(xy) + \int_S f(s)d\mu(s)f(x\sigma(y)) - 2f(x)\int_S f(ys)d\mu(s)\Big| \leq 3\delta\|\mu\| \tag{3.11}$$

for all $x, y \in S$. Since, from (3.5), we have $\int_S f(s)d\mu(s) \neq 0$, then the inequality (3.11) can be written as follows

$$|f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \frac{3\delta\|\mu\|}{|\int_S f(s)d\mu(s)|}$$

for all $x, y \in S$, where g is the function defined in Lemma 3.1. Now, by using the same computation used in [6, Theorem 2.2(iii)] we conclude that f, g are solutions of Wilson's functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y) \tag{3.12}$$

for all $x, y \in S$. By replacing x by t in (3.12) and integrating the result with respect to t we get $\int_S f(ty)d\mu(t)+\int_S f(t\sigma(y))d\mu(t) = 2g(y)\int_S f(t)d\mu(t)$. Since $f \circ \sigma = f$ then we get

$$\begin{aligned}\int_S f(ty)d\mu(t) &+ \int_S f(t\sigma(y))d\mu(t)\\ &= \int_S f(yt)d\mu(t) + \int_S f(y\sigma(t))d\mu(t) = 2f(y)\int_S g(t)d\mu(t).\end{aligned}$$

Then we have $f(y)\int_S g(t)d\mu(t) = g(y)\int_S f(t)d\mu(t)$. So, $g = \frac{\int_S g(t)d\mu(t)}{\int_S f(t)d\mu(t)} f$. For all $x, y \in S$ we have

$$\begin{aligned}(3.13)\quad &\int_S f(xyt)d\mu(t) + \int_S f(x\sigma(y)t)d\mu(t)\\ &= \int_S [f(xty) + f(xt\sigma(y))]d\mu(t) = 2\int_S f(xt)d\mu(t)g(y) = 2\beta f(x)f(y),\end{aligned}$$

where $\beta = \frac{(\int_S g(t)d\mu(t))^2}{\int_S f(t)d\mu(t)}$. Using this in (3.1) we obtain $|2(\beta-1)f(y)f(x)| \le \delta$ for all $x, y \in S$. Since f is assumed to be unbounded then we deduce that $\beta = 1$ and then from (3.13) we deduce that f is a solution of (1.4). This completes the proof. □

3.1. Superstability of the integral Kannappan's functional equation (1.4) on monoids

If S is a monoid (a semigroup with identity element e) then by elementary computations we verify that the superstability of the integral Kannappan's functional equation follows from the superstability of d'Alembert's functional equation (1.2).

Proposition 3.3. *Let M be a topological monoid. Let σ be an involutive anti-automorphism of M and let μ a complex measure with compact support. Let $\delta > 0$ be fixed. If a continuous function $f\colon M \to \mathbb{C}$ satisfies the inequality*

$$(3.14)\qquad \Big|\int_M f(xyt)d\mu(t) + \int_M f(x\sigma(y)t)d\mu(t) - 2f(x)f(y)\Big| \le \delta$$

for all $x, y \in M$, then either f is bounded or f is a solution of the integral Kannappan's functional equation (1.4).

Proof. Let f be an unbounded continuous function which satisfies (3.14). Taking $y = e$ in (3.14) we get

$$\Big|\int_M f(xt)d\mu(t) - f(e)f(x)\Big| \leq \frac{\delta}{2} \tag{3.15}$$

for all $x \in M$. Since f is unbounded then $f(e) \neq 0$, because if $f(e) = 0$ the functions $(x,y) \mapsto \int_M f(xyt)d\mu(t)$; $(x,y) \mapsto \int_M f(x\sigma(y)t)d\mu(t)$ are bounded and from (3.14) and the triangle inequality we get f a bounded function on M. This contradicts the assumption that f is unbounded. Now, from (3.14), (3.15) and the triangle inequality we obtain

$$\begin{aligned}
&|f(e)f(xy) + f(e)f(x\sigma(y)) - 2f(x)f(y)| \\
&\leq \Big|f(e)f(xy) - \int_M f(xyt)d\mu(t)\Big| + \Big|f(e)f(x\sigma(y)) - \int_M f(x\sigma(y)t)d\mu(t)\Big| \\
&+ \Big|\int_M f(xyt)d\mu(t) + \int_M f(x\sigma(y)t)d\mu(t) - 2f(x)f(y)\Big| \leq \frac{\delta}{2} + \frac{\delta}{2} + \delta = 2\delta.
\end{aligned}$$

This inequality can be written as follows

$$\Big|f(xy) + f(x\sigma(y)) - 2f(x)\frac{f(y)}{f(e)}\Big| \leq \frac{2\delta}{|f(e)|}, \quad x, y \in M.$$

From [6, Theorem 2.2(iii)] we deduce that f, g are solutions of Wilson's functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)\frac{f(y)}{f(e)}$$

for all $x, y \in M$, then from [31] f is central. So,

$$\begin{aligned}
\int_M f(xyt)d\mu(t) + \int_M f(x\sigma(y)t)d\mu(t) &= \int_M f(txy)d\mu(t) + \int_M f(tx\sigma(y))d\mu(t) \\
&= \int_M [f(tx(y)) + f((tx)\sigma(y))]d\mu(t) \\
&= 2\frac{f(y)}{f(e)}\int_M f(tx)d\mu(t) = 2\frac{f(y)}{f(e)}\int_M f(xt)d\mu(t).
\end{aligned}$$

Using the equality above with (3.14) after computation we get

$$\Big|f(y)\Big(f(x)-\frac{\int_M f(xt)d\mu(t)}{f(e)}\Big)\Big|\le\frac{\delta}{2}$$

for all $x, y \in M$. Since f is unbounded then $f(x)=\frac{\int_M f(xt)d\mu(t)}{f(e)}$ for all $x \in M$. Thus, for all $x, y \in M$ we get

$$\int_M f(xyt)d\mu(t)+\int_M f(x\sigma(y)t)d\mu(t)=2\frac{f(y)}{f(e)}\int_M f(xt)d\mu(t)=2f(x)f(y).$$

That is: f satisfies the integral Kannappan's functional equation (1.4). This completes the proof. □

4. Solutions of the functional equation (1.5)

The solutions of the functional equation (1.5) with σ an involutive anti-automorphism are explicitly obtained by Elqorachi [13] on semigroups not necessarily abelian in terms of multiplicative functions. In this section we express the solutions of (1.5) where σ is an involutive automorphism in terms of multiplicative functions. The following lemma is obtained in [13] for the case where σ is an involutive anti-automorphism. It still holds for the case where σ is an involutive automorphism.

Lemma 4.1. *Let $\sigma\colon S\to S$ be a morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i \in I$. Let f be a non-zero solution of equation* (1.5)*. Then for all $x \in S$ we have*

$$f(x)=-f(\sigma(x)),$$

$$\int_S f(t)d\mu(t)\neq 0,$$

$$\int_S\int_S f(ts)d\mu(t)d\mu(s)=\int_S\int_S f(\sigma(t)s)d\mu(t)d\mu(s)=0,$$

$$\int_S\int_S f(x\sigma(t)s)d\mu(t)d\mu(s)=f(x)\int_S f(t)d\mu(t),$$

$$\int_S \int_S f(xts)d\mu(t)d\mu(s) = -f(x)\int_S f(t)d\mu(t), \tag{4.1}$$

$$\int_S f(\sigma(x)t)d\mu(t) = \int_S f(xt)d\mu(t).$$

The function defined by

$$g(x) := \frac{\int_S f(xt)d\mu(t)}{\int_S f(s)d\mu(s)} \quad \textit{for } x \in S$$

is a non-zero solution of d'Alembert's functional equation (1.2). *Furthermore,*

$$\int_S \int_S g(ts)d\mu(t)d\mu(s) \neq 0; \quad \int_S g(s)d\mu(s) = 0.$$

That is $Jf = g \in \mathcal{B}$, *where* J *and* $\mathcal{B}$ *are the function and the set defined in Theorem 4.2.*

Theorem 4.2. *Let* $\sigma\colon S \to S$ *be an involutive morphism of* S. *Let* μ *be a complex measure that is a linear combination of Dirac measures* $(\delta_{z_i})_{i\in I}$, *such that* z_i *is in the center of* S *for all* $i \in I$. *Let* $\mathcal{B}$ *consists of the solutions* $g : S \to \mathbb{C}$ *of d'Alembert's functional equation* (1.2) *such that* $\int_S g(t)d\mu(t) = 0$ *and* $\int_S \int_S g(st)d\mu(s)d\mu(t) \neq 0$. *Let* $\mathcal{V}$ *consists of the non-zero solutions of the extension of Van Vleck's functional equation* (1.5). *Then the function* $J\colon \mathcal{V} \to \mathcal{B}$ *defined by*

$$Jf(x) = \frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)}, \quad x \in S, \tag{4.2}$$

is a bijection of $\mathcal{V}$ *onto* $\mathcal{B}$. *In particular* $J(\mathcal{V}) = \mathcal{B}$.

Proof. From Lemma 4.1 the formula (4.2) makes sense, and we have $g := Jf \in \mathcal{B}$ for any $f \in \mathcal{V}$.

Injection: Let f_1 and f_2 be two non-zero solutions of (1.5). If $Jf_1 = Jf_2$ then we get

$$\int_S f_2(t)d\mu(t)\int_S f_1(xt)d\mu(t) = \int_S f_1(t)d\mu(t)\int_S f_2(xt)d\mu(t) \tag{4.3}$$

for all $x \in S$. Since f_1 and f_2 are solutions of (1.5), we have

$$\int_S f_1(x\sigma(y)t)d\mu(t) - \int_S f_1(xyt)d\mu(t) = 2f_1(x)f_1(y) \tag{4.4}$$

and

$$\int_S f_2(x\sigma(y)t)d\mu(t) - \int_S f_2(xyt)d\mu(t) = 2f_2(x)f_2(y).$$

By multiplying (4.4) by $\int_S f_2(t)d\mu(t)$ and using (4.3) we get

$$2f_1(x)f_1(y)\int_S f_2(t)d\mu(t) = 2f_2(x)f_2(y)\int_S f_1(t)d\mu(t). \tag{4.5}$$

By replacing y by s in (4.5) and integrating the result with respect to s we get

$$2f_1(x)\int_S f_1(s)d\mu(s)\int_S f_2(t)d\mu(t) = 2f_2(x)\int_S f_2(s)d\mu(s)\int_S f_1(t)d\mu(t).$$

Since $\int_S f_2(s)d\mu(s)\int_S f_1(t)d\mu(t) \neq 0$, then $f_1 = f_2$.

Surjection: Let $g \in \mathcal{B}$. First we notice that since g is a solution of (1.2) and $\int_S g(s)d\mu(s) = 0$ then if we let $y = s$ in (1.2) and integrating the result with respect to s we deduce that $\int_S g(x\sigma(s))d\mu(s) = -\int_S g(xs)d\mu(s)$. We may define $f\colon S \to \mathbb{C}$ by

$$\begin{aligned} f(x) &= \frac{1}{2}\Big(\int_S g(x\sigma(s))d\mu(s) - \int_S g(xs)d\mu(s)\Big) \\ &= \int_S g(x\sigma(s))d\mu(s) = -\int_S g(xs)d\mu(s). \end{aligned}$$

For all $x, y \in S$ we have

$$\begin{aligned} &\int_S f(x\sigma(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) \\ &= \int_S\int_S g(x\sigma(y)t\sigma(s))d\mu(t)d\mu(s) - \int_S\int_S g(xyt\sigma(s))d\mu(t)d\mu(s) \\ &= \int_S\int_S g(xt\sigma(ys))d\mu(t)d\mu(s) + \int_S\int_S g(xtys)d\mu(t)d\mu(s) \\ &= 2\int_S g(xt)d\mu(t)\int_S g(ys)d\mu(s) = 2f(x)f(y). \end{aligned}$$

Furthermore,

$$\int_S f(s)d\mu(s) = \int_S \int_S g(s\sigma(t))d\mu(s)d\mu(t) = -\int_S \int_S g(st)d\mu(s)d\mu(t) \neq 0.$$

Thus, we get $f \neq 0$. On the other hand, for all $x \in S$ we have

$$\begin{aligned} Jf(x) = \frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)} &= \frac{\int_S \int_S g(xt\sigma(s))d\mu(t)d\mu(s)}{\int_S \int_S g(t\sigma(s))d\mu(t)d\mu(s)} \\ &= \frac{\int_S \int_S g(xt\sigma(s))d\mu(t)d\mu(s) + \int_S \int_S g(xt\sigma(s))d\mu(t)d\mu(s)}{2\int_S \int_S g(t\sigma(s))d\mu(t)d\mu(s)} \\ &= \frac{2g(x)\int_S \int_S g(t\sigma(s))d\mu(t)d\mu(s)}{2\int_S \int_S g(t\sigma(s))d\mu(t)d\mu(s)} = g(x). \end{aligned}$$

This completes the proof. $\square$

In [13] we use [31, Proposition 8.14] to derive the form of the solutions of (1.5) where σ is an involutive anti-automorphism of S. This reasoning no longer works for the present conditions. We will use an elementary approach which works for both situations.

Theorem 4.3. *Let $\sigma\colon S \to S$ be a morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i \in I$. The non-zero central solutions of the integral Van Vleck's functional equation* (1.5) *are the functions of the form*

$$f = \frac{\chi - \chi\circ\sigma}{2}\int_S \chi(\sigma(t))d\mu(t),$$

where $\chi : S \to \mathbb{C}$ is a multiplicative function such that $\int_S \chi(t)d\mu(t) \neq 0$ and $\int_S \chi(\sigma(t))d\mu(t) = -\int_S \chi(t)d\mu(t)$.

Furthermore, if S is a topological semigroup and $\sigma : S \to S$ is continuous, then the non-zero solution f of equation (1.5) *is continuous if and only if χ is continuous.*

Proof. Let f be a non-zero solution of (1.5). Replacing x by xs in (1.5) and integrating the result with respect to s we get

$$(4.6) \qquad \int_S \int_S f(xs\sigma(y)t)d\mu(s)d\mu(t) - \int_S \int_S f(xsyt)d\mu(s)d\mu(t)$$

$$= 2f(y)\int_S f(xs)d\mu(s) = \int_S\int_S f(x\sigma(y)st)d\mu(s)d\mu(t)$$
$$- \int_S\int_S f(xyst)d\mu(s)d\mu(t).$$

By using (4.1) equation (4.6) can be written as follows

$$-f(x\sigma(y)) + f(xy) = 2f(y)g(x),\ x,y\in S, \tag{4.7}$$

where g is the function defined in Lemma 4.1. If we replace y by ys in (1.5) and integrate the result with respect to s we get

$$\int_S\int_S f(x\sigma(y)\sigma(s)t)d\mu(s)d\mu(t) - \int_S\int_S f(xyst)d\mu(s)d\mu(t)$$
$$= 2f(x)\int_S f(ys)d\mu(s).$$

By using (4.1) we obtain that

$$f(x\sigma(y)) + f(xy) = 2f(x)g(y), \quad x,y\in S. \tag{4.8}$$

Adding (4.8) and (4.7) we get that the pair f, g satisfies the sine addition law

$$f(xy) = f(x)g(y) + f(y)g(x) \quad \text{for all } \ x,y\in S.$$

Now, in view of [12, Lemma 3.4], [31, Theorem 4.1] g is abelian. Since g is a non-zero solution of d'Alembert's functional equation (1.2) there exists a non-zero multiplicative function $\chi\colon S\to\mathbb{C}$ such that $g = \frac{\chi+\chi\circ\sigma}{2}$. The rest of the proof is similar to the one used in [13]. This completes the proof. □

Corollary 4.4. *Let S be a semigroup, let σ be an involutive automorphism of S. Let g be a solution of d'Alembert's functional equation* (1.2). *If there exists a complex measure μ that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i\in I$ and $\int_S g(t)d\mu(t) = 0$, $\int_S\int_S g(ts)d\mu(t)d\mu(t) \neq 0$, then there exists a non-zero multiplicative function $\chi\colon S\to\mathbb{C}$ such that $g = \frac{\chi+\chi\circ\sigma}{2}$.*

Proof. Let $g\colon S \to \mathbb{C}$ be a non-zero function which satisfies the conditions of Corollary 4.4. From Theorem 4.2 there exists a non-zero solution of the integral Van Vleck's functional equation (1.5) such that $Tf = g$. From the proof of Theorem 4.3, we get that g is an abelian solution of d'Alembert's functional equation (1.2). That is there exists a non-zero multiplicative function $\chi\colon S \to \mathbb{C}$ such that $g = \frac{\chi + \chi\circ\sigma}{2}$. This completes the proof. $\square$

5. The superstability of the integral Van Vleck's functional equation (1.5)

In the present section we prove the superstability theorem of the integral Van Vleck's functional equation (1.5) on semigroups. First, we prove the following useful lemma.

Lemma 5.1. *Let σ be an involutive morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i \in I$. Let $\delta > 0$ be fixed. If $f : S \to \mathbb{C}$ is an unbounded function which satisfies the inequality*

$$\Big|\int_S f(x\sigma(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) - 2f(x)f(y)\Big| \leq \delta \tag{5.1}$$

for all $x, y \in S$, then, for all $x \in S$

$$f(\sigma(x)) = -f(x), \tag{5.2}$$

$$|\int_S\int_S f(x\sigma(s)t)d\mu(s)d\mu(t) - f(x)\int_S f(t)d\mu(t)| \leq \frac{\delta}{2}\|\mu\|, \tag{5.3}$$

$$|\int_S\int_S f(xst)d\mu(s)d\mu(t) + f(x)\int_S f(t)d\mu(t)| \leq \frac{3\delta\|\mu\|}{2}, \tag{5.4}$$

$$\int_S f(t)d\mu(t) \neq 0, \tag{5.5}$$

$$\int_S\int_S f(st)d\mu(s)d\mu(t) = 0, \tag{5.6}$$

$$|\int_S f(xs)d\mu(s) - \int_S f(\sigma(x)s)d\mu(s)| \leq \frac{4\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{5.7}$$

The function g defined by

$$g(x)=\frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)} \quad \text{for } x\in S \tag{5.8}$$

is unbounded on S and satisfies the following inequality

$$|g(xy)+g(x\sigma(y))-2g(x)g(y)|\leq \frac{3\delta\|\mu\|^2}{(|\int_S f(s)d\mu(s)|)^2} \quad \text{for all } x,y\in S. \tag{5.9}$$

Furthermore,

(i) $\int_S g(t)d\mu(t)=0,\ \int_S\int_S g(ts)d\mu(t)d\mu(t)\neq 0,$

(ii) *g is an abelian solution of d'Alembert's functional equation (1.2) and $Jf=g\in\mathcal{B}$,*

(iii) *f,g are solutions of Wilson's functional equation*

$$f(xy)+f(x\sigma(y))=2f(x)g(y) \tag{5.10}$$

for all $x,y\in S$.

Proof. Equation (5.2): Replacing y by $\sigma(y)$ in (5.1) we get

$$\Big|\int_S f(xyt)d\mu(t)-\int_S f(x\sigma(y)t)d\mu(t)-2f(x)f(\sigma(y))\Big|\leq\delta \tag{5.11}$$

for all $x,y\in S$. By adding the result of (5.1) and (5.11) and using the triangle inequality we obtain $|2f(x)(f(y)+f(\sigma(y)))|\leq 2\delta$ for all $x\in S$. Since f is assumed to be unbounded we get (5.2).

Inequality (5.3): By replacing x by $\sigma(s)$ in (5.1) and integrating the result with respect to s we have

$$\begin{aligned}\Big|\int_S\int_S f(\sigma(s)\sigma(y)t)d\mu(s)d\mu(t)-\int_S\int_S f(\sigma(s)yt)d\mu(s)d\mu(t)&\\ -2f(y)\int_S f(\sigma(s))d\mu(s)\Big|&\leq\delta\|\mu\|\end{aligned} \tag{5.12}$$

for all $y\in S$. By using (5.2) we have

$$\int_S\int_S f(\sigma(s)\sigma(y)t)d\mu(s)d\mu(t)=-\int_S\int_S f(y\sigma(t)s)d\mu(s)d\mu(t)$$

and $\int_S f(\sigma(s))d\mu(s) = -\int_S f(s)d\mu(s)$. So, equation (5.12) can be written as follows:

$$\Big| -\int_S\int_S f(y\sigma(s)t)d\mu(s)d\mu(t) - \int_S\int_S f(y\sigma(s)t)d\mu(s)d\mu(t) + 2f(y)\int_S f(s)d\mu(s)\Big| \le \delta\|\mu\|$$

for all $y \in S$. This proves (5.3).

Inequality (5.4): Taking $y = s$ in (5.1) and integrating the result with respect to s we get

$$\Big| \int_S\int_S f(x\sigma(s)t)d\mu(s)d\mu(t) - \int_S\int_S f(xst)d\mu(s)d\mu(t) - 2f(x)\int_S f(s)d\mu(s)\Big| \le \delta\|\mu\|$$

for all $x \in S$. Note that

$$\begin{aligned} &\Big| \int_S\int_S f(xst)d\mu(s)d\mu(t) + f(x)\int_S f(s)d\mu(s)\Big| \\ &= \Big| \int_S\int_S f(xst)d\mu(s)d\mu(t) + 2f(x)\int_S f(s)d\mu(s) - \int_S\int_S f(x\sigma(s)t)d\mu(s)d\mu(t) \\ &\qquad + \int_S\int_S f(x\sigma(s)t)d\mu(s)d\mu(t) - f(x)\int_S f(s)d\mu(s)\Big|. \end{aligned}$$

So, by using (5.3), (5.12) and the triangle inequality we deduce (5.4).

Condition (5.5): Since f is assumed to be an unbounded solution of the inequality (5.1) then $f \neq 0$. Now assume that $\int_S f(t)d\mu(t) = 0$. Replacing x by xs in (5.1) and integrating the result with respect to s and using the fact that μ is concentrated in the center of S we get

$$\Big| \int_S\int_S f(x\sigma(y)st)d\mu(s)d\mu(t) - \int_S\int_S f(xyst)d\mu(s)d\mu(t) - 2f(y)\int_S f(xs)d\mu(s)| \le \delta\|\mu\|$$

for all $x, y \in S$. Note that

$$\begin{aligned}
2f(y)\int_S f(xs)d\mu(s) &= 2f(y)\int_S f(xs)d\mu(s) \\
&+ \int_S\int_S f(xyst)d\mu(s)d\mu(t) - \int_S\int_S f(x\sigma(y)st)d\mu(s)d\mu(t) \\
&- \Big(\int_S\int_S f(xyst)d\mu(s)d\mu(t) + f(xy)\int_S f(s)d\mu(s)\Big) \\
&+ \int_S\int_S f(x\sigma(y)st)d\mu(s)d\mu(t) + f(x\sigma(y))\int_S f(s)d\mu(s) \\
&+ f(xy)\int_S f(s)d\mu(s) - f(x\sigma(y))\int_S f(s)d\mu(s).
\end{aligned}$$

So, by using (5.4), (5.1), $\int_S f(t)d\mu(t) = 0$ and the triangle inequality we get that $y \mapsto f(y)\int_S f(xs)d\mu(s)$ is a bounded function on S. Since f is unbounded then we obtain $\int_S f(xs)d\mu(s) = 0$ for all $x \in S$. By applying this to (5.1) we get f a bounded function on S and this contradicts the assumption that f is an unbounded function. So, we have (5.5).

Inequality (5.9): By similar computation as above the function g defined by (5.8) is an unbounded function on S. Furthermore,

$$\begin{aligned}
&\int_S f(s)d\mu(s)\int_S f(k)d\mu(k)[g(xy) + g(x\sigma(y)) - 2g(x)g(y)] \\
&= \int_S f(k)d\mu(k)\int_S f(xyt)d\mu(t) + \int_S f(s)d\mu(s)\int_S f(x\sigma(y)t)d\mu(t) \\
&\quad - 2\int_S f(xs)d\mu(s)\int_S f(ys)d\mu(s) \\
&= \int_S \Big[f(xyt)\int_S f(k)d\mu(k) + \int_S\int_S f(xytks)d\mu(k)d\mu(s)\Big]d\mu(t) \\
&+ \int_S \Big[f(x\sigma(y)t)\int_S f(s)d\mu(s) - \int_S\int_S f(x\sigma(y)t\sigma(k)s)d\mu(k)d\mu(s)\Big]d\mu(t) \\
&+ \int_S\int_S \Big[\int_S f(xs\sigma(yk)t)d\mu(t) - \int_S f(xsykst)d\mu(t) - 2f(xs)f(yk)\Big]d\mu(k)d\mu(s).
\end{aligned}$$

So, by using (5.3), (5.4) and (5.1) we get

$$\Big|\int_S f(s)d\mu(s)\int_S f(k)d\mu(k)\Big[g(xy)+g(x\sigma(y))-2g(x)g(y)\Big]\Big|$$
$$\leq \frac{3\delta\|\mu\|^2}{2}+\frac{\delta\|\mu\|^2}{2}+\delta\|\mu\|^2=3\delta\|\mu\|^2,$$

which proves (5.9).

Equation (5.6): Since g is unbounded, from [6] g satisfies d'Alembert's functional equation (1.2). From (5.3), (5.4) and the triangle inequality we have

$$|\int_S\int_S f(x\sigma(s)t)d\mu(s)d\mu(t)+\int_S f(xst)d\mu(s)d\mu(t)|\leq 2\delta\|\mu\| \tag{5.13}$$

for all $x,y\in S$. Since $g=\frac{\int_S f(xk)d\mu(k)}{\int_S f(t)d\mu(t)}$, the inequality (5.13) can be written as follows

$$\Big|\int_S f(k)d\mu(k)\int_S g(x\sigma(k))d\mu(k)+\int_S f(k)d\mu(k)\int_S g(xk)d\mu(k)\Big|\leq 2\delta\|\mu\|.$$

On the other hand g is a solution of d'Alembert's functional equation (1.2), then we get $|2g(x)\int_S g(k)d\mu(k)|\leq\frac{2\delta\|\mu\|}{|\int_S f(k)d\mu(k)|}$ for all $x\in S$. Since g is unbounded, we deduce that $\int_S g(k)d\mu(k)=0$. That is $\int_S\int_S f(st)d\mu(s)d\mu(t)=0$, which proves (5.6).

Inequality (5.7): By replacing x by sk in (5.1), integrating the result with respect to s and k and using (5.6) and the fact that μ is concentrated in the center of S we obtain

$$\Big|\int_S\int_S f(\sigma(y)skt)d\mu(s)d\mu(k)d\mu(t)$$
$$-\int_S\int_S f(yskt)d\mu(s)d\mu(k)d\mu(t)\Big|\leq\delta\|\mu\|^2 \tag{5.14}$$

for all $y\in S$. Note that

$$\int_S\int_S\int_S f(\sigma(y)skt)d\mu(s)d\mu(k)d\mu(t)-\int_S\int_S\int_S f(yskt)d\mu(s)d\mu(k)d\mu(t)$$
$$=\int_S\int_S\int_S f(\sigma(y)skt)d\mu(s)d\mu(k)d\mu(t)+\int_S f(\sigma(y)t)d\mu(t)\int_S f(s)d\mu(s)$$

$$-\Big(\int_S\int_S\int_S f(yskt)d\mu(s)d\mu(k)d\mu(t)+\int_S f(yt)d\mu(t)\int_S f(s)d\mu(s)\Big)$$

$$-(\int_S f(\sigma(y)t)d\mu(t)-\int_S f(yt)d\mu(t))\int_S f(s)d\mu(s).$$

So from (5.14), (5.4) and the triangle inequality we get

$$|\int_S f(\sigma(y)t)d\mu(t)-\int_S f(yt)d\mu(t)|\leq\frac{4\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}.$$

This proves (5.7).

Equation (5.10): From (5.1), (5.3), (5.4) and the triangle inequality we get

$$(5.15)\quad \Big|\int_S f(s)d\mu(s)f(xy)+\int_S f(s)d\mu(s)f(x\sigma(y))-2f(x)\int_S f(ys)d\mu(s)\Big|$$

$$\leq\Big|\int_S f(s)d\mu(s)f(xy)+\int_S\int_S f(xyst)d\mu(s)d\mu(t)\Big|$$

$$+\Big|\int_S f(s)d\mu(s)f(x\sigma(y))-\int_S\int_S f(x\sigma(y)st)d\mu(s)d\mu(t)|Big|$$

$$+\Big|\int_S\int_S f(x\sigma(y)st)d\mu(s)d\mu(t)-\int_S\int_S f(xyst)d\mu(s)d\mu(t)$$

$$-2f(x)\int_S f(ys)d\mu(s)\Big|\leq 3\delta\|\mu\|$$

for all $x,y\in S$. Since from (5.5) we have $\int_S f(s)d\mu(s)\neq 0$, then the inequality (5.15) can be written as follows

$$|f(xy)+f(x\sigma(y))-2f(x)g(y)|\leq\frac{3\delta\|\mu\|}{|\int_S f(s)d\mu(s)|}$$

for all $x,y\in S$, where g is the function defined in Lemma 5.1. Now, by using the same computation as used in [6, Theorem 2.2(iii)] we conclude that f,g are solutions of Wilson's functional equation (5.10). This completes the proof. □

Now, we are ready to prove the main result of the present section.

THEOREM 5.2. *Let σ be an involutive morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that z_i is in the center of S for all $i \in I$. Let $\delta > 0$ be fixed. If $f : S \to \mathbb{C}$ satisfies the inequality*

$$\Big|\int_S f(x\sigma(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) - 2f(x)f(y)\Big| \leq \delta \tag{5.16}$$

for all $x, y \in S$, then either f is bounded and $|f(x)| \leq \frac{\|\mu\|+\sqrt{\|\mu\|^2+2\delta}}{2}$ for all $x \in S$, or f is a solution of the integral Van Vleck's functional equation (1.5).

PROOF. Assume that f is an unbounded solution of (5.16). From Lemma 5.1(iii) f, g are solutions of Wilson's functional equation (5.10). Taking $y = s$ in (5.10) and integrating the result with respect to s we get

$$\int_S f(xs)d\mu(s) + \int_S f(x\sigma(s))d\mu(s) = 0 \tag{5.17}$$

because $\int_S g(s)d\mu(s) = 0$. By replacing y by $s\sigma(k)$ in in (5.10) and integrating the result with respect to s and k we obtain

$$\begin{aligned}
&\int_S\int_S f(xs\sigma(k))d\mu(s)d\mu(k) + \int_S\int_S f(x\sigma(s)k)d\mu(s)d\mu(k)\\
&\qquad = 2f(x)\int_S\int_S g(s\sigma(k))d\mu(s)d\mu(k)\\
&\qquad = \int_S\int_S f(xs\sigma(k))d\mu(s)d\mu(k) + \int_S\int_S f(xk\sigma(s))d\mu(s)d\mu(k)\\
&\qquad = 2\int_S\int_S f(xs\sigma(k))d\mu(s)d\mu(k).
\end{aligned}$$

That is

$$\int_S\int_S f(xs\sigma(k))d\mu(s)d\mu(k) = f(x)\int_S\int_S g(s\sigma(k))d\mu(s)d\mu(k). \tag{5.18}$$

Now from (5.3) and (5.18) we get

$$\Big|f(x)\Big(\int_S\int_S g(s\sigma(k))d\mu(s)d\mu(k) - \int_S f(t)d\mu(t)\Big)\Big| \leq \frac{\delta\|\mu\|}{2}$$

for all $x \in S$. Since f is assumed to be unbounded, we get

$$\int_S \int_S g(s\sigma(k))d\mu(s)d\mu(k) = \int_S f(t)d\mu(t). \tag{5.19}$$

The function g satisfies d'Alembert's equation (1.2) and $\int_S g(s)d\mu(s) = 0$, then we have $\int_S g(yk)d\mu(k) = -\int_S g(y\sigma(k))d\mu(k)$ for all $y \in S$. So, by using the definition of g, equations (5.18) and (5.19) we have

$$\int_S g(yk)d\mu(k) = -\int_S g(y\sigma(k))d\mu(k) = \frac{-\int_S \int_S f(y\sigma(k)t)d\mu(k)d\mu(t)}{\int_S f(s)d\mu(s)} \tag{5.20}$$

$$= \frac{-f(y)\int_S \int_S g(\sigma(k)t)d\mu(k)d\mu(t)}{\int_S f(s)d\mu(s)} = \frac{-f(y)\int_S f(t)d\mu(t)}{\int_S f(s)d\mu(s)} = -f(y).$$

Finally, from (5.10), (5.17) and (5.20) for all $x, y \in S$ we have

$$\int_S f(x\sigma(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) = -\int_S f(x\sigma(y)\sigma(t))d\mu(t)$$

$$-\int_S f(xyt)d\mu(t) = -\int_S [f(x\sigma(yt)) + f(xyt)]d\mu(t)$$

$$= -2f(x)\int_S g(yt)d\mu(t) = 2f(x)f(y).$$

That means f is a solution of Van Vleck's functional equation (1.5). This completes the proof. □

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