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ON A CERTAIN TYPE OF PEXIDER EQUATIONS

Abstract. The present paper deals with general solutions of the following functional equations: $f(xy)=\overline{f(x)}\,\overline{f(y)}$, $f(xy)=\overline{f(x)}+\overline{f(y)}$, $f(xy)=f^T(x)f^T(y)$, $f(xy)=f^T(x)+f^T(y)$, where the symbols on the right-hand sides of these equations denote the conjugate of complex numbers (or quaternions) and the transpose of matrices, respectively.

Let $(X, +)$ be a semigroup and $(Y, +)$ be a group. Let $e\colon Y\to Y$ be an involutive group automorphism, i.e. $e(u+v) = e(u)+e(v)$ and $e(e(u)) = u$ for all $u, v\in Y$. Conjugation in the additive group $(\mathbf{C}, +)$ or in the multiplicative group $(\mathbf{C}\setminus\{0\}, \cdot)$ of complex numbers, matrix transition in the additive group of $n\times n$-matrices are examples of involutive group automorphisms. J. Tabor [2] considered the following alternative functional equation

$$f(x+y) = f(x)+f(y) \text{ or } f(x+y) = e(f(x)+f(y))$$

for all $x, y\in X$, where $f\colon X\to Y$ is an unknown function.

The equation

$$f(x+y) = e(f(x)+f(y)) \tag{$*$}$$

is a certain type of the Pexider equation. As a particular case of equation $(*)$ J. Tabor cosidered the equation

$$f(x+y)=\overline{f(x)}\cdot\overline{f(y)},$$

where $f\colon X\to\mathbf{C}\setminus\{0\}$ is an unknown function from a group $(X, +)$ into the multiplicative group $(\mathbf{C}\setminus\{0\}, \cdot)$ of all non-zero complex numbers. The general solution of this equation is expressed by means of some real valued homomorphism from $(X, +)$ into $(\mathbf{C}\setminus\{0\}, \cdot)$ and the cube roots of unity.

In this paper we shall consider the functional equations $f(xy)=\overline{f(x)}\,\overline{f(y)}$, $f(xy)=\overline{f(x)}+\overline{f(y)}$, $f(xy)=f^T(x)f^T(y)$, $f(xy) = f^T(x)+f^T(y)$, where f is an unknown function defined on some algebraic structures. These equations can

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be considered as some type of Pexider equations. The general solutions of the above equations involve a real valued homomorphism and the cube roots of unity (in some cases there are other solutions).

1. We begin with some general considerations connected with the Pexider equation.

The Pexider equation on groupoids G_1 and G_2 is said to be the functional equation

$$f(xy) = g(x)h(y) \tag{1}$$

for $x, y \in G_1$, where $f, g, h\colon G_1 \to G_2$ are unknown functions. A triple of functions (f, g, h) satisfying equation (1) will be called a solution of Pexider equation (1).

A groupoid G is said to be a *group with zero* if there exists an element $0 \in G$ such that $G^* = G \setminus \{0\}$ is a group and $0x = x0 = 0$ for all $x \in G$.

THEOREM 1. *Let G_1 be a groupoid with identity, and let G_2 be a group with zero. A triple of functions $f, g, h\colon G_1 \to G_2$ is a solution of Pexider equation* (1) *if and only if it has one of the following forms:*

(A) $\qquad f(x) = a_1\varphi(x)a_2, \quad g(x) = a_1\varphi(x), \quad h(x) = \varphi(x)a_2$

for $x \in G_1$, where $\varphi\colon G_1 \to G_2$ is a homomorphism from the groupoid G_1 with identity to the group G_2 with zero, and $a_1, a_2 \in G_2 \setminus \{0\}$ are constants;

(B) $f = 0$ *and $g, h\colon G_1 \to G_2$ are arbitrary functions such that $g(x) = 0$ or $h(x) = 0$ for every $x \in G_1$.*

Proof. Let a triple of functions (f, g, h) be a solution of Pexider equation (1). Put $a_1 = g(1)$ and $a_2 = h(1)$. Consider the following two cases:

(i) $a_1 \neq 0$ and $a_2 \neq 0$,

(ii) $a_1 = 0$ or $a_2 = 0$.

Case (i). Put $\varphi(x) = a_1^{-1} g(x)$ for $x \in G_1$. Note that $f(x) = g(1)\,h(x) = a_1 h(x)$ and $f(x) = g(x)h(1) = g(x)a_2$ for $x \in G_1$. Hence $f(x) = a_1(a_1^{-1} g(x))a_2 = a_1\varphi(x)a_2$ for $x \in G_1$. Thus $f(x) = a_1\,\varphi(x)a_2$, $g(x) = a_1\,\varphi(x)$, $h(x) = \varphi(x)a_2$ for $x \in G_1$. It is easy to check that $\varphi\colon G_1 \to G_2$ is a homomorphism. Thus the triple (f, g, h) has form (A).

Case (ii). Note that if condition (ii) is satisfied then $f = 0$. Hence $g(x) = 0$ or $h(x) = 0$ for all $x \in G_1$. The triple (f, g, h) has form (B).

It is easy to verify that any triple of functions (f, g, h) having form (A) or (B) is a solution of Pexider equation (1).

In the sequel we shall use the following.

COROLLARY 1. *Let G_1 be a groupoid with identity, and let G_2 be a group with zero. If a triple of functions $f, g, h\colon G_1 \to G_2$ is a solution of Pexider equation* (1) *such that $g(1), h(1) \in G_2^*$, then $f(x) = a_1\,\varphi(x)\,a_2$, $g(x) = a_1\,\varphi(x)$, $h(x) = \varphi(x)a_2$ for $x \in G_1$, where $a_1 = g(1)$, $a_2 = h(1)$ and $\varphi\colon G_1 \to G_2$ is a homomorphism from the groupoid G_1 with identity to the group G_2 with zero.*

This corollary results immediately from the construction of the solution of Pexider equation (1) applied in the proof of Theorem 1.

2. Let G be a groupoid and let $\mathbf{C}$ denote the set of all complex numbers. Consider the functional equation

$$(2) \qquad f(xy) = \overline{f(x)}\,\overline{f(y)}$$

for all $x, y \in G$, where $f: G \to \mathbf{C}$ is an unknown function. The symbol $\overline{f(x)}$ denotes here the complex conjugate of $f(x)$.

THEOREM 2. *Let G be a groupoid with identity. The general solution of functional equation* (2) *has the form*

$$(3) \qquad f(x) = a\varphi(x)$$

for $x \in G$, where $\varphi: G \in \mathbf{R}$ is a homomorphism from the groupoid G with identity to the multiplicative semigroup $\mathbf{R}$ of real numbers, $a \in \mathbf{C}$ and $a^3 = 1$.

Proof. Let a function $f: G \to \mathbf{C}$ be a solution of equation (2). If $f(1) = 0$ then $f(x) = 0$ for $x \in G$. Thus f is of form (3), where $\varphi: G \to \mathbf{R}$ is a zero homomorphism.

Suppose that $f(1) = a \neq 0$. By equation (2) we get $a = \bar{a}^2$. It is easy to check that $a = \bar{a}^2$ iff $a^3 = 1$. It follows from Corollary 1 that the function $\varphi: G \to \mathbf{C}$ defined by $\varphi(x) = \bar{a}^{-1}\overline{f(x)}$ for $x \in G$ is a homomorphism from the groupoid G to the multiplicative semigroup $\mathbf{C}$ of complex numbers. We shall show that $\varphi(x) \in \mathbf{R}$ for $x \in G$. From Corollary 1 we get $\underline{f(x)} = \bar{a}^2\,\underline{\varphi(x)}$ and so $f(x) = a\,\varphi(x)$ for $x \in G$. Hence $\varphi(x) = \bar{a}^{-1}\overline{f(x)} = \bar{a}^{-1}\overline{\bar{a}\,\varphi(x)} = \overline{\varphi(x)}$ for $x \in G$. Thus the function f is of form (3).

It can be easily verified that each function f of form (3) satisfies equation (2).

Let G be a groupoid. Consider the functional equation

$$(4) \qquad f(xy) = \overline{f(x)} + \overline{f(y)}$$

for all $x, y \in G$, where $f: G \to \mathbf{C}$ is an unknown function.

THEOREM 3. *Let G be a groupoid with identity. A function $f: G \to \mathbf{C}$ is a solution of equation* (4) *if and only if f is a homomorphism from the groupoid G with identity to the additive group $\mathbf{R}$ of all real numbers.*

Proof. Suppose that a function $f: G \to \mathbf{C}$ satisfies equation (4). We have $f(1) = \overline{f(1)} + \overline{f(1)}$, whence $f(1) = 0$. By (4) we get $f(x) = \overline{f(x)}$ for $x \in G$. Moreover, $f(xy) = f(x) + f(y)$ for $x, y \in G$.

THEOREM 4. *Let G be a semigroup. The general solution $f: G \to \mathbf{C}$ of functional equation* (2) *has the form*

$$(5) \qquad f(x) = b\varphi(x)$$

for $x \in G$, where $\varphi: G \to \mathbf{R}$ is a homomorphism from the semigroup G to the multiplicative semigroup $\mathbf{R}$ of real numbers, $b \in \mathbf{C}$ and $b^3 = 1$.

Proof. Let a function $f: G \to \mathbf{C}$ be a solution of equation (2). Put $G_1 = \{x \in G: f(x) \neq 0\}$ and $G_2 = G \setminus G_1$. Note that if $G_1 \neq \varnothing$ (resp. $G_2 \neq \varnothing$), then G_1 (resp. G_2) is a subsemigroup of the semigroup G. Furthermore, $G_1 G_2 \subset G_2$ and $G_2 G_1 \subset G_2$.

If $G_1 = \varnothing$ then f is of form (5), where $\varphi: G \to \mathbf{R}$ is a zero homomorphism. Assume that $G_1 \neq \varnothing$. For an arbitrary element $x \in G_1$ we have $f(x) = \frac{f(x)}{\overline{f(x)}}\overline{f(x)}$. Put $k(x) = \frac{f(x)}{\overline{f(x)}}$ for $x \in G_1$. Suppose that $x, y, z \in G_1$. Then $f(x(yz)) = \overline{f(x)}\, f(y)\, f(z)$ and $f((xy)z) = f(x)\, f(y)\overline{f(z)}$. Hence $\frac{f(x)}{\overline{f(x)}} = \frac{f(z)}{\overline{f(z)}}$ for $x, z \in G_1$. Thus $k(x) = \text{const} = a$ for $x \in G_1$, where $a \in \mathbf{C}$ and $a \neq 0$. Moreover, we have $f(x) = a\overline{f(x)}$ for $x \in G_1$. Note that $a\bar{a} = 1$. We define a function $\tilde{\varphi}: G_1 \to \mathbf{C}^*$ by the formula $\tilde{\varphi}(x) = a^{-2} f(x)$, $x \in G_1$. The function $\tilde{\varphi}$ is a homomorphism from the semigroup G_1 to the multiplicative group $\mathbf{C}^*$ of all non-zero complex numbers. Indeed, $\tilde{\varphi}(xy) = a^{-2} f(xy) = a^{-2}\overline{f(x)}\,\overline{f(y)} = a^{-4} f(x)\, f(y) = \tilde{\varphi}(x)\,\tilde{\varphi}(y)$ for all $x, y \in G_1$. Observe that $\overline{f(x)} = a\tilde{\varphi}(x)$ for $x \in G_1$. From the above equalities we get $f(x) = \bar{a}\,\overline{\tilde{\varphi}(x)}$ and so $a^2\tilde{\varphi}(x) = \bar{a}\,\overline{\tilde{\varphi}(x)}$ for $x \in G_1$. Hence $a^3\tilde{\varphi}(x) = \overline{\tilde{\varphi}(x)}$ for $x \in G_1$. For $x, y \in G_1$ we have $a^3\tilde{\varphi}(xy) = \overline{\tilde{\varphi}(xy)}$, $a^3\tilde{\varphi}(x)\tilde{\varphi}(y) = \overline{\tilde{\varphi}(x)}\,\overline{\tilde{\varphi}(y)}$, $\overline{\tilde{\varphi}(x)}\,\tilde{\varphi}(y) = \overline{\tilde{\varphi}(x)}\,\overline{\tilde{\varphi}(y)}$ and so $\tilde{\varphi}(y) = \overline{\tilde{\varphi}(y)}$ for all $y \in G_1$. Hence $\tilde{\varphi}$ maps G_1 into $\mathbf{R}^*$. Since $a^3\tilde{\varphi}(x) = \tilde{\varphi}(x)$, $x \in G_1$ we get $a^3 = 1$. Define a function $\varphi: G \to \mathbf{R}$ by the formula

$$\varphi(x) = \begin{cases} \tilde{\varphi}(x) & \text{for } x \in G_1, \\ 0 & \text{for } x \in G_2. \end{cases}$$

It is not difficult to check that the function φ is a homomorphism from the semigroup G to the multiplicative semigroup $\mathbf{R}$ of real numbers. Thus $f(x) = a^2\varphi(x)$ for $x \in G$. Put $b = a^2$. Observe that $b^3 = 1$. Then f has form (5).

It is easy to see that any function of form (5) satisfies equation (2).

THEOREM 5. *Let G be a semigroup. A function $f: G \to \mathbf{C}$ is a solution of equation* (4) *if and only if f is a homomorphism from the semigroup G to the additive group $\mathbf{R}$ of real numbers.*

P r o o f. Let $f: G \to \mathbf{C}$ be a solution of equation (4). We have $f(x) = (f(x) - \overline{f(x)}) + \overline{f(x)}$ for $x \in G$. Put $k(x) = f(x) - \overline{f(x)}$ for $x \in G$. Note that $f(x(yz)) = \overline{f(x)} + f(y) + f(z)$ and $f((xy)z) = f(x) + f(y) + \overline{f(z)}$ for $x, y, z \in G$. Hence $f(x) - \overline{f(x)} = f(z) - \overline{f(z)}$ for all $x, z \in G$ and so $k(x) = \text{const} = a$, $a \in \mathbf{C}$. We get $f(x) = a + \overline{f(x)}$ for $x \in G$. Since f satisfies equation (4) we obtain $f(xy) = a + \overline{f(xy)} = a + f(x) + f(y)$ and $f(xy) = \overline{f(x)} + \overline{f(y)} = -2a + f(x) + f(y)$ for $x, y, \in G$ and so $a = 0$. Hence $f(x) = \overline{f(x)}$ for $x \in G$. Therefore $f: G \to \mathbf{R}$ is a homomorphism from the semigroup G to the additive group $\mathbf{R}$ of real numbers.

REMARK 1. Let the groupoid G be the additive group $\mathbf{R}$ of real numbers. Taking into account Theorem 2 and [1, Theorem 13.1.4] we get that a function $f: \mathbf{R} \to \mathbf{C}$ is a continuous solution of equation (2) if and only if it has one of the forms

$$f = 0,$$
$$f(x) = a\mathrm{e}^{cx} \quad \text{for } x \in \mathbf{R},$$
where $a \in \mathbf{C}$, $c \in \mathbf{R}$ are constants and $a^3 = 1$.

REMARK 2. Let the groupoid G be the multiplicative semigroup $\mathbf{R}$ of real numbers. By Theorem 4 and [1, Theorem 13.1.6] we infer that a function $f: \mathbf{R} \to \mathbf{C}$ is a continuous solution of equation (2) if and only if it has one of the forms
$$f = 0,$$
$$f = a,$$
$$f(x) = a|x|^c,$$
$$f(x) = a|x|^c \operatorname{sgn} x, \quad x \in \mathbf{R},$$
where $a \in \mathbf{C}$, $c \in \mathbf{R}^+$ are constants and $a^3 = 1$.

REMARK 3. Let the groupoid G be the additive group of real numbers. Taking into account Theorem 3 and [1, Theorem 5.4.2] we obtain that a function $f: \mathbf{R} \to \mathbf{C}$ is a continuous solution of equation (4) if and only if it has the form $f(x) = cx$, $x \in \mathbf{R}$, where $c \in \mathbf{R}$ is a constant.

REMARK 4. Let the groupoid G be the multiplicative group $\mathbf{R}^*$ of all non-zero real numbers. In virtue of Theorem 3 and [1, Theorem 13.1.5] we infer that a function $f: \mathbf{R}^* \to \mathbf{C}$ is a continuous solution of equation (4) if and only if it has the form $f(x) = c\ln|x|$, $x \in \mathbf{R}^*$, where $c \in \mathbf{R}$ is a constant.

3. Let G be a groupoid and let $\mathbf{H}$ denote the set of quaternions. Consider the functional equation

$$f(xy) = \overline{f(x)}\,\overline{f(y)} \tag{6}$$

for $x, y \in G$ where $f: G \to \mathbf{H}$ is an unknown function. The symbol $\overline{f(x)}$ denotes here the quaternion conjugate of $f(x)$.

THEOREM 6. *Let G be a groupoid with identity. The general solution of functional equation* (6) *has the form*

$$f(x) = q\varphi(x) \tag{7}$$

for $x \in G$, where $\varphi: G \to \mathbf{R}$ is a homomorphism from the groupoid G with identity to the multiplicative semigroup $\mathbf{R}$ of real numbers, $q \in \mathbf{H}$ and $q^3 = 1$.

Proof. Let a function $f: G \to \mathbf{H}$ be a solution of equation (6). If $f(1) = 0$ then $f(x) = 0$ for $x \in G$. Thus f has form (7), where $\varphi: G \to \mathbf{R}$ is a zero homomorphism.

Now, suppose that $f(1) = q \neq 0$. By (6) we get $q = \bar{q}^2$. It is easy to check that $q = \bar{q}^2$ iff $q^3 = 1$. It follows from Corollary 1 that the function $\varphi: G \to \mathbf{H}$ defined by the formula $\varphi(x) = \bar{q}^{-1}\overline{f(x)}$, $x \in G$, is a homomorphism from the groupoid G to the multiplicative semigroup $\mathbf{H}$ of quaternions. We shall show that φ is a real valued function. Note that $\bar{q}^{-1} = q$. From Corollary 1 we have

$$\varphi(x) = q\overline{f(x)},$$
$$\varphi(x) = \overline{f(x)}\,q,$$
$$\varphi(x) = q\,f(x)q$$

for $x \in G$. One can verify that $\alpha\beta = \beta\alpha$ iff $\alpha\overline{\beta} = \overline{\beta}\alpha$ for all α, $\beta \in \mathbf{H}$. From the above equalities we get $\varphi(x) = q^2 f(x)$ for $x \in G$. Hence $\overline{\varphi(x)} = \overline{f(x)}\,\bar{q}^2 = \overline{f(x)}\,q = \varphi(x)$ for $x \in G$. Thus $\overline{f(x)} = \bar{q}\varphi(x)$ whence $f(x) = q\varphi(x)$ for $x \in G$, i.e. the function f is of form (7).

It is not difficult to check that every function f of form (7) satisfies equation (6).

REMARK 5. *The quaternion $q \in \mathbf{H}$ such that $q^3 = 1$ has the form $q = 1$ or $q = -\frac{1}{2} + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, where b, c, $d \in \mathbf{R}$ and $b^2 + c^2 + d^2 = \frac{3}{4}$.*

Let G be a groupoid. Consider the functional equation

$$f(xy) = \overline{f(x)} + \overline{f(y)} \tag{8}$$

for x, $y \in G$, where $f: G \to \mathbf{H}$ is an unknown function.

THEOREM 7. *Let G be a groupoid with identity. A function $f: G \to \mathbf{H}$ is a solution of equation* (8) *if and only if f is a homomorphism from the groupoid G with identity to the additive group $\mathbf{R}$ of real numbers.*

We omit the easy proof.

4. Let $GL(2, \mathbf{R})$ be the full linear group of square matrices of order 2 over the real field $\mathbf{R}$. The mapping

$$\mathbf{C}^* \ni a + bi \to \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in GL(2, \mathbf{R})$$

is an isomorphic embedding of the multiplicative group $\mathbf{C}^*$ of all non-zero complex numbers into the full linear group $GL(2, \mathbf{R})$. Thus we can regard $\mathbf{C}^*$ and $\mathbf{R}^*$ as subsets of $GL(2, \mathbf{R})$. Let $S \subset GL(2, \mathbf{R})$ be the set of all symmetric matrices, i.e. $A \in S$ iff $A = A^T$, where A^T is the transpose of A.

Let G be a groupoid. Consider the functional equation

$$f(xy) = f^T(x)\,f^T(y) \tag{9}$$

for all x, $y \in G$, where $f: G \to GL(2, \mathbf{R})$ is an unknown function. The symbol $f^T(x)$ denotes the transpose of the martix $f(x)$. It turns out that equation (9) has also solutions of a form different from that occurring in the preceding cases.

THEOREM 8. *Let G be a groupoid with identity. A function $f: G \to GL(2, \mathbf{R})$ is a solution of equation* (9) *if and only if f has one of the following forms:*

(A) *$f(x) = \psi(x)$ for $x \in G$, where $\psi: G \to GL(2, \mathbf{R})$ is a homomorphism from the groupoid G with identity to the full linear group $GL(2, \mathbf{R})$ such that $\psi(G) \subset S$;*

(B) *$f(x) = a\varphi(x)$ for $x \in G$, where $\varphi: G \to \mathbf{R}^*$ is a homomorpism from the groupoid G with identity to the multiplicative group $\mathbf{R}^*$ of all non-zero real numbers, $a \in \mathbf{C}$ and $a^3 = 1$.*

Proof. Assume that a function $f: G \to GL(2, \mathbf{R})$ is a solution of equation (9). Put

$$f(1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GL(2, \mathbf{R}).$$

Since $f(1) = f^T(1) f^T(1)$, we have $A = (A^2)^T$. The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

are the only ones satisfying the condition $A = (A^2)^T$.

Suppose that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $f(x) = f^T(1) f^T(x) = f^T(x)$ for $x \in G$. It is enough to take $\psi(x) = f(x)$ for $x \in G$, to get (A).

Now, suppose that

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Note that $(A^T)^{-1} = A$. It is not difficult to check that $AX = XA$ iff $X \in C^*$ and $AX^T = X^T A$ iff $X \in \mathbf{C}^*$ for an arbitrary $X \in GL(2, \mathbf{R})$. It follows from Corollary 1 that the function $\varphi: G \to GL(2, \mathbf{R})$ defined by the formula

$$\varphi(x) = (A^T)^{-1} f^T(x), \quad x \in G,$$

is a homomorphism.

We shall show that φ has all its values in $\mathbf{R}^*$. By virtue of Corollary 1 we have

$$\varphi(x) = A f^T(x),$$
$$\varphi(x) = f^T(x) A,$$
$$\varphi(x) = A f(x) A$$

for $x \in G$. It follows from the above equalities that $f(x), f^T(x) \in \mathbf{C}^*$ for $x \in G$. Hence $\varphi(x) = A^2 f(x)$ for $x \in G$. Furthermore, $\varphi^T(x) = (A^2)^T f^T(x) = A f^T(x) = \varphi(x)$ for $x \in G$. Since $\varphi(x) \in \mathbf{C}^*$, we get $\varphi^T(x) = \overline{\varphi(x)}$ and so $\varphi(x) \in \mathbf{R}^*$ for an arbitrary element $x \in G$. Moreover, $f^T(x) = A^T \varphi(x)$ then $f(x) = A\varphi(x)$ for $x \in G$. The case where

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

is quite analogous. Then the function f is of form (B).

It is easy to verify that every function of form (A) or (B) satisfies equation (9).

Let G be a groupoid. Let $M(2, \mathbf{R})$ be the group of all square real matrices of order 2 under matrix addition. Consider the functional equation

$$f(xy) = f^T(x) + f^T(y) \tag{10}$$

for all $x, y \in G$, where $f: G \to M(2, \mathbf{R})$ is an unknown function.

THEOREM 9. *Let G be a groupoid with identity. A function $f: G \to M(2, \mathbf{R})$ is a solution of equation* (10) *if and only if f is a homomorphism from the groupoid G with identity to the additive group $M(2, \mathbf{R})$ such that $f(G) \subset S$.*

We omit an easy proof of this theorem.

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