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A NOTE ON THE INCREASE OF THE OPERATIONAL FUNCTION

$$\exp\left(\lambda \sum_{k=1}^{n} A_k s^{p_k}\right)$$

Abstract. The main result of the paper is the estimation of the operational function of the exponential type which is the parametric one in the sense of the Mikusinski's theory. The estimation may be applied in studying the behaviour of solutions of partial differential equations.

Introduction. The exponential functions

$$\exp\left(\lambda \sum_{k=1}^{m} A_k s^{p_k}\right) \quad \text{for } \lambda \in \mathbf{R}, \tag{1}$$

where $A_k \in \mathbf{R}$, $0 < p_k < 1$ and s denotes the differential operator, play an important role in the theory of partial differential equations and some types of convolution equations on the real halfline. If we know the behaviour of the exponential function (1) at infinity we can indicate an uniqueness class of solutions of the Cauchy problem for some differential equations. In accordance with the notation of the operational calculus for the exponential function (1) we also use an other symbol $F_{p_1,\ldots,p_m}(\lambda, t)$when $\lambda > 0$ and coefficients A_k satisfy some additional conditions. In this case the exponential function (1) can be represented by means of the Laplace inversian integral

$$F_{p_1,\ldots,p_m}(\lambda, t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left[zt - \lambda \sum_{k=1}^{m} A_k z^{p_k}\right] dz.$$

Using the theory of contour integrals we shall show the more general inequalities to the inequality which was considered in [2], related to the exponential function $\exp(-s^{\alpha}\lambda)$.

Main theorems.

THEOREM 1. *Let*

$$F_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left[zt - \lambda\left(\sum_{j=1}^{m} B_j z^{p_j} + \sum_{k=1}^{n} C_k z^{q_k}\right)\right] dz$$

for $t, \lambda > 0$, *where* $1 > p_1 > \ldots > p_m > 0$ *and* $1 > q_1 > \ldots > q_n > 0$, $p_1 > q_1$ *and* $p_j \neq q_k$ *for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$; *moreover, let* $B_j > 0$ *and* $C_k < 0$ *for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$, *respectively. If* $0 < B_1 \leqslant 1$, $\varepsilon \in (0, 1 - 2^{-p_1})$ *and*

1° $\lambda \cdot t^{-1} > [p_1 B_1(1-\varepsilon)]^{-1}$,

2° $\lambda \cdot t^{-p_1} > 2[p_1^5 B_1]^{-1}$,

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$$3^\circ\ \lambda\cdot t^{-1} > [p_1(1-\varepsilon)B_1]^{-1}\left[-\sum_{k=1}^{n}\frac{C_k\cos\left(q_k\cdot\frac{\pi}{2}\right)}{\varepsilon B_1\cos\left(p_1\cdot\frac{\pi}{2}\right)}\right]^{\frac{1-p_1}{p_1-q_1}}$$

then the following inequality

$$0 < F_{p_1,\dots,p_m,\ q_1,\dots,q_n}(\lambda,t) <$$

$$< (\lambda p_1)^{\frac{1}{1-p_1}} t^{-\frac{1}{1-p_1}}[(1-\varepsilon)B_1]^{-\frac{1}{p_1}}\exp\{-(1-p_1)\cdot[\lambda(1-\varepsilon)B_1 p_1^{p_1}]^{\frac{1}{1-p_1}} t^{-\frac{p_1}{1-p_1}}\}$$

holds.

Proof. In the beginning we will give some elementary lemmas.

LEMMA 1. *If* $\lambda\in\left(0,\frac{\pi}{2}\right)$ *and* $p\in(0,1)$ *then*

$$\sin(p\cdot\lambda)\cdot[\sin\lambda]^{-1} < \sin\left(\lambda\cdot\frac{\pi}{2}\right).$$

LEMMA 2. *If* $p\in(0,1)$ *and* $\lambda \geqslant 2p^{-4}$ *then*

$$\lambda^{\frac{1}{p}-1}\cdot\exp\left[-\lambda\cos\left(p\cdot\frac{\pi}{2}\right)\right] < \exp\left[-\lambda(1-p)\right].$$

LEMMA 3. *If* $p\in(0,1)$, $\lambda>0$, $t>0$, $q\in(0,1)$, *and* $\lambda\cdot t^{-p} > 2\cdot(qp^5)^{-1}$ *then*

$$(\lambda\cdot t^{-p})^{\frac{1}{p(1-p)}} > (2p^{-3})^{\frac{1}{p(1-p)}}.$$

LEMMA 4. *If* $p\in(0,1)$ *then*

$$[\pi\cdot p(1-p)]^{-1}\left(\frac{1}{2}p^3\right)^{\frac{1}{p(1-p)}} < \frac{1}{2}p^{\frac{1}{1-p}}.$$

LEMMA 5. *If* $\lambda\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $p_1, p_2\in(0,1)$, $p_1>p_2$, *and* $P(\lambda)=\cos(p_2\lambda)\cdot$ $\cdot[\cos(p_1\lambda)]^{-1}$ *then*

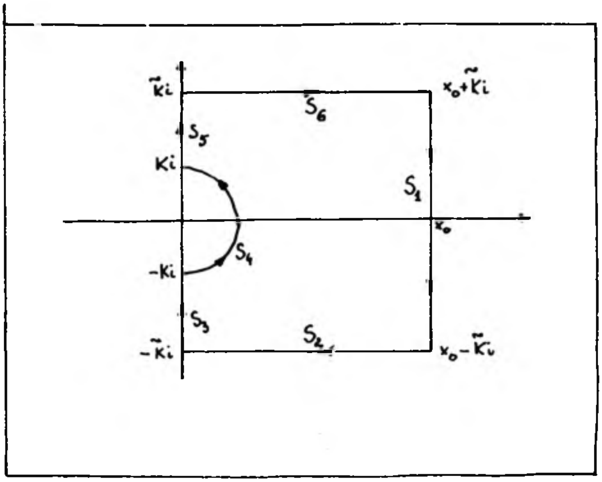
$$\max_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]} P(\lambda) = \cos\left(p_2\cdot\frac{\pi}{2}\right)\cdot\left[\cos\left(p_1\cdot\frac{\pi}{2}\right)\right]^{-1}.$$

We shall denote by $\tilde{F}_{p_1,\dots,p_m}(\lambda,t)$ the integrand of the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\exp\left[zt-\lambda\left(\sum_{k=1}^{m}A_k z^{p_i}\right)\right]dz$$

and as usual we shall take this branch of the complex function z^p which takes real values on the real axis. Let us consider the contour S consisting of segments

$S_1 = [x_0+\tilde{K}i, x_0-\tilde{K}i]$, $S_2 = [x_0-\tilde{K}i, -\tilde{K}i]$, $S_3 = [-\tilde{K}i, Ki]$, $S_5 = [Ki, \tilde{K}i]$, $S_6 = [\tilde{K}i, x_0+\tilde{K}i]$ where $K, \tilde{K}, x_0 \in \mathbf{R}$ and $K < \tilde{K} < x_0$ and of semicircumference S_4 of radius K lying on the right of the imaginary axis.



The integral of $\tilde{F}_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t)$ taken along S is null. It is easy to show that if K increases infinitely, the moduli of the integral of $\tilde{F}_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t)$ taken along S_2 and S_6 tend to zero. It proves that we may transform the formula of $F_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t)$, changing the path of integration from the imaginary axis to the path $S_3 \cup S_4 \cup S_5$ and the radius K may be chosen arbitrarily, as convenient for estimation. When K increase infinitely, S_3 transforms to $(-\infty, -Ki]$ and S_5 to $[Ki, \infty)$.

$F_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t)$ is a real and positive function [3] when $\lambda > 0$ and $t > 0$, since it is the convolution of real and positive functions (see [1]). At first we show the estimation of the modulus of the integral of $\tilde{F}_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t)$ along S_4. Making the parametrization of the semicircumference S_4: $z(u) = K\exp(iu)$ we have

$$|J_1| = \left|\frac{1}{2\pi i}\int_{S_4} \exp\left[zt-\lambda\left(\sum_{k=1}^{m} B_k z^{p_k} + \sum_{j=1}^{n} C_j z^{q_j}\right)\right] dz\right| \leqslant$$

$$\leqslant \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left\{Kt\cos u - \lambda\left[\sum_{k=1}^{m} B_k K^{p_k}\cos(p_k u) + \sum_{j=1}^{n} C_j K^{q_j}\cos(q_j u)\right]\right\} du.$$

It follows from the assumptions that, for $k = 2, \ldots, m$, $B_k > 0$. Hence

$$|J_1| \leqslant \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left\{Kt\cos u - \lambda\left[B_1 K^{p_1}\cos(p_1 u) + \sum_{j=1}^{n} C_j K^{q_j}\cos(q_j u)\right]\right\} du =$$

$$= \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\{Kt\cos u -$$

$$-\lambda B_1 K^{p_1}\cos(p_1 u)\left[1 + \sum_{j=1}^{n} C_j B_1^{-1} K^{q_j - p_1}\cdot\cos(q_j u)\cos^{-1}(p_1 u)\right]\} du.$$

It follows from the assumptions and Lemma 5 that, for $j = 1, \ldots, n$ and $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\cos(q_j u)\cos^{-1}(p_1 u) < \cos\left(q_j \frac{\pi}{2}\right)\cdot\cos^{-1}\left(p_1\cdot\frac{\pi}{2}\right)$$

and then

$$|J_1| \leqslant \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left\{Kt\cos u - \lambda B_1 K^{p_1}\cos(p_1 u)\left[1 + \sum_{j=1}^{n} C_j B_1^{-1} K^{q_j - p_1}\cdot\right.\right.$$

$$\left.\left.\cdot\cos\left(q_j\frac{\pi}{2}\right)\cos^{-1}\left(p_1\cdot\frac{\pi}{2}\right)\right]\right\} du.$$

Let $K > 1$. It follows from the assumptions that, for $j = 1, \ldots, n$, $K^{q_j - p_1} < K^{q_1 - p_1}$. Hence

$$|J_1| \leqslant \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left\{Kt\cos u - \lambda B_1 K^{p_1}\cos(p_1 u)\left[1 + K^{q_1 - p_1}\sum_{j=1}^{n} C_j B_1^{-1}\cdot\right.\right.$$

$$\left.\left.\cdot\cos\left(q_j\cdot\frac{\pi}{2}\right)\cos^{-1}\left(p_1\cdot\frac{\pi}{2}\right)\right]\right\} du.$$

Let us fix $\varepsilon \in (0, 1 - 2^{-p_1})$. If $K > 1$ and

$$K > \left\{-\left[\varepsilon B_1\cos\left(p_1\cdot\frac{\pi}{2}\right)\right]^{-1}\sum_{j=1}^{n} C_j\cos\left(q_j\cdot\frac{\pi}{2}\right)\right\}^{\frac{1}{p_1 - q_1}}$$

then

$$-K^{q_1-p_1}\left[B_1\cos\left(p_1\cdot\frac{\pi}{2}\right)\right]^{-1}\sum_{j=1}^{n}C_j\cos\left(q_j\cdot\frac{\pi}{2}\right)<\varepsilon$$

and hence

$$|J_1|\leqslant\frac{K}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp\left[Kt\cos u-\lambda B_1K^{p_1}\cos(p_1u)(1-\varepsilon)\right]du.$$

Let us choose $K=\lambda^{\frac{1}{1-p_1}}(tw)^{-\frac{1}{1-p_1}}$; then

$$|J_1|\leqslant\frac{K}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp\left\{\lambda^{\frac{1}{1-p_1}}(wt^{p_1})^{-\frac{1}{1-p_1}}\left[\cos u-wB_1(1-\varepsilon)\cos(p_1u)\right]\right\}du.$$

Let us denote $A_w(u)=\cos u-wB_1(1-\varepsilon)\cos(p_1u)$; then $A'_w(u)=-\sin u+$ $+wB_1(1-\varepsilon)p_1\sin(p_1u)$. If

$$0<w\leqslant\left[B_1(1-\varepsilon)p_1\sin\left(p_1\cdot\frac{\pi}{2}\right)\right]^{-1}$$

then for $u\in\left(-\frac{\pi}{2},0\right)$ the function $A'_w(u)$ is positive and for $u\in\left(0,\frac{\pi}{2}\right)$ is negative, which follows from Lemma 1. Hence, if w satisfies the upper inequality, the function $A_w(u)$ has maximum at the point $u=0$ and $A_w(0)=1-wB_1(1-\varepsilon)$. Then

$$|J_1|\leqslant\frac{K}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp\left\{\lambda^{\frac{1}{1-p_1}}(wt^{p_1})^{-\frac{1}{1-p_1}}\left[1-wB_1(1-\varepsilon)\right]\right\}du=$$

$$=\frac{K}{2}\exp\left\{\lambda^{\frac{1}{1-p_1}}t^{-\frac{p_1}{1-p_1}}w^{-\frac{1}{1-p_1}}\left[1-wB_1(1-\varepsilon)\right]\right\}.$$

Let for $0<w\leqslant\left[B_1(1-\varepsilon)p_1\sin\left(p_1\cdot\frac{\pi}{2}\right)\right]^{-1}$

$$B(w)=w^{-\frac{1}{1-p_1}}\left[1-wB_1(1-\varepsilon)\right];$$

then $w_0=\left[p_1B_1(1-\varepsilon)\right]^{-1}$ is the point of minimum of the function $B(w)$ and

$$B(w_0)=-p_1^{\frac{p_1}{1-p_1}}(1-p_1)\cdot\left[B_1(1-\varepsilon)\right]^{\frac{1}{1-p_1}}.$$

Putting w_0 we get the best estimation of $|J_1|$.

$$|J_1| \leqslant \frac{1}{2}[\lambda \cdot p_1 B_1(1-\varepsilon)]^{\frac{1}{1-p_1}} t^{-\frac{1}{1-p_1}} \exp\{-(1-p_1) p_1^{\frac{p_1}{1-p_1}} \cdot$$

$$\cdot [\lambda B_1(1-\varepsilon)]^{\frac{1}{1-p_1}} t^{-\frac{p_1}{1-p_1}}\}.$$

If $B_1 \in (0,1]$ and $\varepsilon \in (0, 1-2^{-p_1})$ then $[(1-\varepsilon)B_1] < 1$ and $[(1-\varepsilon)B_1]^{\frac{1}{1-p_1}} <$ $< [(1-\varepsilon)B_1]^{-\frac{1}{p_1}}$.

Hence

$$(2) \qquad |J_1| < \frac{1}{2}(\lambda p_1)^{\frac{1}{1-p_1}} t^{-\frac{1}{1-p_1}} [B_1(1-\varepsilon)]^{-\frac{1}{p_1}} \exp\{-(1-p_1) \cdot$$

$$\cdot p_1^{\frac{p_1}{1-p_1}} [\lambda B_1(1-\varepsilon)]^{\frac{1}{1-p_1}} t^{-\frac{p_1}{1-p_1}}\}.$$

And it is the half of the right side of the inequality in the theorem. Putting $w_0 = [p_1(1-\varepsilon)B_1]^{-1}$ we get

$$K_0(\lambda, t) = \lambda^{\frac{1}{1-p_1}} (w_0 t)^{-\frac{1}{1-p_1}}.$$

If we shall take λ, t fulfilling the inequalities 1° and 3° (the assumptions of the theorem) then by simple calculatuon we obtain the following inequalities

$$(3) \qquad \begin{gathered} K_0(\lambda, t) > 1 \\ K_0(\lambda, t) > \left\{-\left[\varepsilon B_1 \cos\left(p_1 \cdot \frac{\pi}{2}\right)\right]^{-1} \sum_{j=1}^{n} C_j \cos\left(q_j \cdot \frac{\pi}{2}\right)\right\}^{\frac{1}{p_1 - q_1}} \end{gathered}$$

This implies that inequality (2) is true when the inequalities 1° and 3° hold.

Now we will find the estimation along the segment S_5. Identical estimation we can give for the segment S_3. Let us parametrize the halfline $[K_0(\lambda, t), \infty)$ in the form $z(u) = iu$. Hence

$$|J_2| = \left|\frac{1}{2\pi i} \int_{S_5} \exp[zt - \lambda(\sum_{k=1}^{m} B_k z^{p_k} + \sum_{j=1}^{n} C_j z^{q_j})] \, dz\right| \leqslant$$

$$\leqslant \frac{1}{2\pi} \int_{K_0(\lambda,t)}^{\infty} \exp\left\{-\lambda\left[\sum_{k=1}^{m} B_k u^{p_k} \cos\left(p_k \cdot \frac{\pi}{2}\right) + \sum_{j=1}^{n} C_j u^{q_j} \cos\left(q_j \cdot \frac{\pi}{2}\right)\right]\right\} du.$$

It follows from the similar transformations given in the first part of the proof that

$$|J_2| \leqslant \frac{1}{2\pi} \int_{K_0(\lambda,t)}^{\infty} \exp\left[-\lambda B_1 u^{p_1} \cos\left(p_1 \cdot \frac{\pi}{2}\right) \cdot (1-\varepsilon)\right] du.$$

Changing the variable $y = \lambda(1-\varepsilon)B_1 u^{p_1}$ we have

$$|J_2| \leqslant \frac{1}{2\pi p_1}[\lambda(1-\varepsilon)B_1]^{-\frac{1}{p_1}} \int\limits_{[\lambda(1-\varepsilon)B_1 K_0(\lambda,t)^{p_1}]}^{\infty} y^{\frac{1}{p_1}-1}\cdot\exp\left[-y\cos\left(p_1\cdot\frac{\pi}{2}\right)\right]\mathrm{d}y.$$

If $0 < B_1 \leqslant 1$ and $\varepsilon\in(0, \ 1-2^{-p_1})$ and $\lambda\cdot t^{-p_1} > 2(p_1^5 B_1)^{-1}$ then $\lambda(1-\varepsilon)B_1 K_0(\lambda, t)^{p_1} > 2p_1^{-4}$. It follows from Lemma 2 that

$$|J_2| < \frac{1}{2\pi p_1}[\lambda(1-\varepsilon)B_1]^{-\frac{1}{p_1}} \int\limits_{[\lambda(1-\varepsilon)B_1 K_0(\lambda,t)^{p_1}]}^{\infty} \exp[-y(1-p_1)]\,\mathrm{d}y =$$

$$= [2\pi p_1(1-p_1)]^{-1}[\lambda(1-\varepsilon)B_1]^{-\frac{1}{p_1}}\exp\{-(1-p_1)p_1^{\frac{p_1}{1-p_1}}\cdot$$

$$\cdot[B_1(1-\varepsilon)]^{\frac{1}{1-p_1}}t^{-\frac{p_1}{1-p_1}}\}.$$

It follows from Lemmas 3 and 4 that

$$|J_2| < \frac{1}{4}(\lambda p_1)^{\frac{1}{1-p_1}}t^{-\frac{1}{1-p_1}}[B_1(1-\varepsilon)]^{-\frac{1}{p_1}}\exp\{-(1-p_1)p_1^{\frac{p_1}{1-p_1}}\cdot$$

$$\cdot[\lambda B_1(1-\varepsilon)]^{\frac{1}{1-p_1}}t^{-\frac{p_1}{1-p_1}}\}.$$

In that way we get the half of the right side of the inequality (2). The similar estimation we may get for the integral along the segment S_3 and it ends the proof.

THEOREM 2. *Let*

$$F_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t) = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} \exp[zt-\lambda(\sum_{j=1}^{m} B_j z^{p_j} + \sum_{k\in 1}^{n} C_k z^{q_k})]\,\mathrm{d}z$$

for $t, \lambda > 0$, where $1 > p_1 > \ldots > p_m > 0$ and $1 > q_1 > \ldots > q_n > 0$, $p_1 > q_1$ and $p_j \neq q_k$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, respectively. If $B_1 > 1$, $\varepsilon\in(0, 1-B_1^{-1})$, and

1° $\lambda\cdot t^{-p_1} > 2p_1^{-5}$,

2° $\lambda\cdot t^{-1} > [p_1 B_1(1-\varepsilon)]^{-1}$,

3° $\lambda\cdot t^{-1} > [p_1(1-\varepsilon)B_1]^{-1}\left\{-\left[\varepsilon B_1\cos\left(p_1\cdot\frac{\pi}{2}\right)\right]^{-1}\cdot\sum_{k=1}^{n} C_k\cos\left(q_k\cdot\frac{\pi}{2}\right)\right\}^{\frac{1-p_1}{p_1-q_1}}$

then the following inequality

$F_{p_1,\ldots,p_m,q_1,\ldots,q_n}(\lambda, t) <$

$$< [p_1\lambda(1-\varepsilon)B_1]^{\frac{1}{1-p_1}}t^{-\frac{1}{1-p_1}}\cdot\exp\{-(1-p_1)p_1^{\frac{p_1}{1-p_1}}[\lambda(1-\varepsilon)B_1]^{\frac{1}{1-p_1}}t^{-\frac{p_1}{1-p_1}}\}$$

holds.

The proof of Theorem 2 is quite similar to the proof of Theorem 1.

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