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ORTHOGONALITY ON THE GIVEN HYPERBOLIC PLANES

Abstract. It has been proved, by K. Menger in [3], that all concepts of the Bolyai-Lobachevsky geometry can be defined in terms of the operations "joining" and "intersecting". In the present paper a similar definition of the orthogonality on hyperbolic plane (being a substitute for the Bolyai -Lobachevsky plane) over a finite field with characteristic different from 2 or over a finite extension of the rational field to a subfield of the real field is given and investigated.

1. **Introduction.** Let F be a finite field with characteristic different from 2 or a rational field, or a finite extension of the rational field to a subfield of the real field (in the two last cases we write shortly "Q-field"). We denote a set of all non-zero squares of F by F^+. F^* denotes a set of all non-zero elements of F, and $F^- := F^* \backslash F^+$. F^- is nonempty in our class of fields.

Let Π be a projective plane over F, and C be a nondegenerate and nonempty (as a set of projective points) conic on Π. A projective lines containing exactly one (resp. zero, two) point lying on C will be called here a *tangent lines* (an *exterior* and *secant lines* respectively). A projective point lying on two (resp. zero) tangent lines will be called an *exterior* (resp. *interior*) *point* [5].

In this paper we consider an incidence structure $(\mathscr{I}, \mathscr{L}_2, \mathscr{L}_0)$, where $\mathscr{I}$ is the interior of C, $\mathscr{L}_2$ (resp. $\mathscr{L}_0$) is a set of non-empty intersections of a secant (resp. exterior) lines with $\mathscr{I}$. This structure may be called a *hyperbolic* (*Bolyai-Lobachevsky*) *plane* over F [4]. The concept of orthogonality is based on the polarity corelation with respect to C which is defined as follows [2]. If C is described on Π by the equation

$$\sum_{i,j=1}^{3} c_{i,j} x_i x_j = 0,$$

where $c_{i,j} = c_{j,i}, c_{i,j}, x_i \in F$ for $i,j = 1,2,3$, then a polar $\Pi_C(p)$ of a point $p \in \Pi$ has the equation

$$\sum_{i,j=1}^{3} c_{i,j} p_i x_j = 0,$$

where $[p_1, p_2, p_3]_\sim$ are the homogeneous coordinates of p. We denote similarly a pole of a line L by $\Pi_C(L)$. Two lines L_1, L_2 are *orthogonal* iff $\Pi_C(L_1) \in L_2$ (it is known that $\Pi_C(L_1) \in L_2$ iff $\Pi_C(L_2) \in L_1$).

The main result is an "incidence interior characterization" of the orthogonality. This characterization is based on a relations (a) and (b) defined as follows.

Let L be a secant or exterior projective line and p be an interior or exterior point not on L. We say that a *pair* (L, p) *satisfies* a *condition* (a) (*resp.* (*b*)) if and

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only if each secant (resp. exterior) line passing through p meets L in an interior point and every lines passing through p and interior points on L are secant (resp. exterior).

2. Basic theorem. Let $[x_1, x_2, x_3]_\sim$, be a homogeneous coordinates of a projective points such that the conic C has a canonical equation

$$\sum_{i=1}^{3} \lambda_i x_i^2 = 0,$$

where $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$ and if F is a **Q**-field then $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 < 0$. Hence the dual conic has equation

$$\sum_{i=1}^{3} \lambda_i^{-1} A_i^2 = 0,$$

where $[A_1, A_2, A_3]_\sim$ are the homogeneous coordinates of a projective lines. We write shortly $p(x_1, x_2, x_3)$ and $L(A_1, A_2, A_3)$ instead of writing a projective point p with coordinates $[p_1, p_2, p_3]_\sim$ and projective line L with coordinates $[A_1, A_2, A_3]_\sim$.

By a simple calculating we have an equivalences

$$p(p_1, p_2, p_3) \text{ is an interior (resp. exterior) point iff}$$

$$-\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i x_i^2 \in F^- (\text{resp. } F^+),$$

and

$$L(A_1, A_2, A_3) \text{ is a secant (resp. exterior) line iff}$$

$$-\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i^{-1} A_i^2 \in F^+ (\text{resp. } F^-).$$

Since $p(p_1, p_2, p_3)$ is a pole of $L(A_1, A_2, A_3)$ iff $\mu A_i = \lambda_i x_i$ for $\mu \in F^*$ and $i = 1, 2, 3$ then

$$L(A_1, A_2, A_3) \text{ and } L(B_1, B_2, B_3) \text{ are orthogonal iff } \sum_{i=1}^{3} \lambda_i^{-1} A_i B_i = 0.$$

The common point of a different projective lines L and M we denote by $L \cdot M$, and the projective line passing through different points p and q we denote by $p \cdot q$.

Given two different lines $L(A_1, A_2, A_3)$ and $M(B_1, B_2, B_3)$ and $L \cdot M$ has coordinates $[x_1, x_2, x_3]_\sim$. By a simple calculating we obtain

$$-\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i x_i^2 \in F^+ \text{ iff } -\Big(\sum_{i=1}^{3} \lambda_i^{-1} A_i^2\Big)\Big(\sum_{i=1}^{3} \lambda_i^{-1} B_i^2\Big) + \Big(\sum_{i=1}^{3} \lambda_i^{-1} A_i B_i\Big)^2 \in F^+.$$

Analogically for two different points $p(p_1, p_2, p_3)$ and $q(q_1, q_2, q_3)$, if $p \cdot q$ has coordinates $[Y_1, Y_2, Y_3]_\sim$ then

$$-\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i^{-1} Y_i^2 \in F^+ \text{ iff } -\Big(\sum_{i=1}^{3} \lambda_i p_i^2\Big)\Big(\sum_{i=1}^{3} \lambda_i q_i^2\Big) + \Big(\sum_{i=1}^{3} \lambda_i p_i q_i\Big)^2 \in F^+.$$

For an arbitrary functions $f = f(x_1, \ldots, x_n)$ and $g = g(x_1, \ldots, x_n)$ we define a relation $\simeq$ as follows

$$f \simeq g \text{ if and only if } f(x_1, \ldots, x_n) \in F^+ \text{ iff } g(x_1, \ldots, x_n) \in F^+$$

for every $x_1, \ldots, x_n \in F$.

Let $L(A_1, A_2, A_3)$ be a projective line, and $p(p_1, p_2, p_3)$ be a projective point neither on L nor on C. Moreover, let $q(q_1, q_2, q_3) \in L$ be an arbitrary projective point, and $p \cdot q = M(Y_1, Y_2, Y_3)$. One can easily verify that the pair (L, p) satisfies condition (a) (resp. (b)) iff

$$(*) \quad \begin{cases} -\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i^{-1} Y_i^2 \simeq -(\sum_{i=1}^{3} \lambda_i p_i^2)(\sum_{i=1}^{3} \lambda_i q_i^2), \\ -\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i q_i^2 \simeq -(\sum_{i=1}^{3} \lambda_i^{-1} A_i^2)(\sum_{i=1}^{3} \lambda_i^{-1} Y_i^2) \end{cases}$$

for every $q \in L$, when $-1 \in F^-$ and L is a secant line or p is an exterior point or $-1 \in F^+$ and L is an exterior line or p is an interior point (resp. $-1 \in F^+$ and L is a secant line or p is an exterior point or $-1 \in F^-$ and L is an exterior line or p is an interior point).

If $p = \Pi_C(L)$ then $(*)$ holds. We shall show that $(*)$ implies $p = \Pi_C(L)$. We assume that $(*)$ holds. Since

$$\begin{cases} -\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i^{-1} Y_i^2 \simeq -(\sum_{i=1}^{3} \lambda_i p_i^2)(\sum_{i=1}^{3} \lambda_i q_i^2) + (\sum_{i=1}^{3} \lambda_i p_i q_i)^2, \\ -\lambda_1\lambda_2\lambda_3 \sum_{i=1}^{3} \lambda_i q_i^2 \simeq -(\sum_{i=1}^{3} \lambda_i^{-1} A_i^2)(\sum_{i=1}^{3} \lambda_i^{-1} Y_i^2) + (\sum_{i=1}^{3} \lambda_i^{-1} A_i Y_i)^2 \end{cases}$$

for every $q \in L$, then

$$(**) \quad \begin{cases} -(\sum_{i=1}^{3} \lambda_i p_i^2)(\sum_{i=1}^{3} \lambda_i q_i^2) \simeq -(\sum_{i=1}^{3} \lambda_i p_i^2)(\sum_{i=1}^{3} \lambda_i q_i^2) + (\sum_{i=1}^{3} \lambda_i p_i q_i)^2, \\ -(\sum_{i=1}^{3} \lambda_i^{-1} A_i^2)(\sum_{i=1}^{3} \lambda_i^{-1} Y_i^2) \simeq -(\sum_{i=1}^{3} \lambda_i^{-1} A_i^2)(\sum_{i=1}^{3} \lambda_i^{-1} Y_i^2) + (\sum_{i=1}^{3} \lambda_i^{-1} A_i Y_i)^2 \end{cases}$$

for every $q \in L$ (i.e. $\sum_{i=1}^{3} q_i A_i = 0$ and $\sum_{i=1}^{3} p_i Y_i = 0$).

Now, we shall prove

LEMMA 1. *If F is a finite field* ($\operatorname{char} F \neq 2$) *and*

$$H(x_1, x_2) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2, \qquad h(x_1, x_2) = a x_1^2 + b x_1 x_2 + c x_2^2,$$

where $\beta, \gamma, b, c, x_1, x_2 \in F$, $\alpha, a \in F^*$, and $H \simeq h$, then there exists $\lambda \in F^*$ such that $H = \lambda^2 h$.

Proof. Let F contains n elements. F^+ contains $\frac{n-1}{2}$ elements, but each of the sets $H(F, 1)$ and $h(F, 1)$ has at least $\frac{n+1}{2}$ elements. Consequently H and h are proportional.

LEMMA 2. *Let F be a* **Q**-*field, H and h be as above, $H \simeq h$, and $H(F,F) \cap \cap F^+ \neq \emptyset, F^- \cap h(F,F) \neq \emptyset$. Then there exists $\lambda \in F^*$ such that $H = \lambda^2 h$.*

Proof. The above problem may be reduced to the following functions

$$\tilde{H}(x) = x^2+\beta x+\gamma \text{ and } \tilde{h}(x) = x^2+c.$$

Now, we have

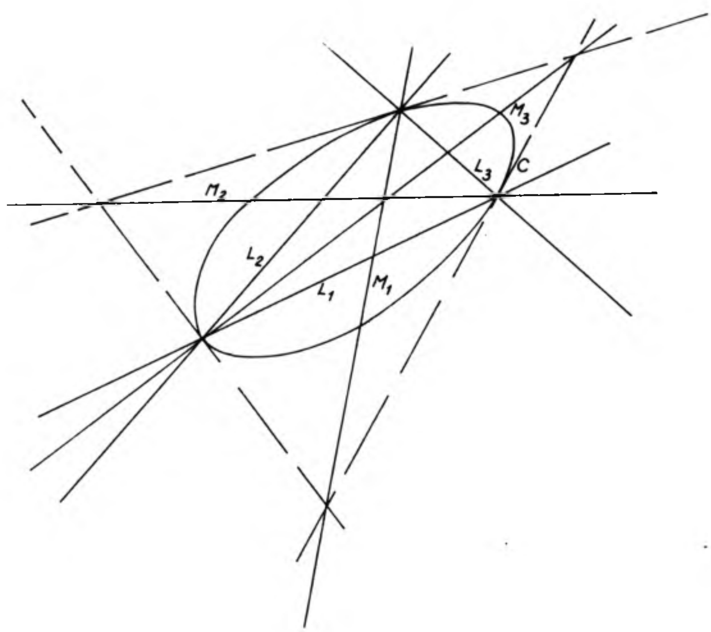
$$\tilde{H}(x) \in F^+ \text{ iff } \left(x = 0 \text{ or } x = \frac{\gamma-z^2}{2z-\beta}, \text{ where } z \in F \backslash \left\{\frac{\beta}{2}\right\}\right),$$

and

$$\tilde{h}(x) \in F^+ \text{ iff } \left(x = 0 \text{ or } x = \frac{c-u^2}{2u}, \text{ where } u \in F^*\right).$$

The condition $\tilde{H} \simeq \tilde{h}$ implies that the equation

$$\frac{c-u^2}{2u} = \frac{\gamma-z^2}{2z-\beta}$$



has the solution for every $u \in F^*$, but it is possible iff $\beta = 0$ and $c = \gamma$, and we have $\tilde{H} = \tilde{h}$. Consequently Lemma 2 holds.

From (**) by using of Lemmas 1 and 2 we obtain

THEOREM. *If L is a secant line or p is an exterior point, then* (L, p) *satisfies* (a) (*resp.* (b)) *iff* $p = \Pi_C(L)$, *when* $-1 \in F^-$ (*resp.* $-1 \in F^+$).

If L is an exterior line or p is an interior point, then (L, p) *satisfies* (b) (*resp.* (a)) *iff* $p = \Pi_C(L)$, *when* $-1 \in F^-$ (*resp.* $-1 \in F^+$).

We say that projective lines L_1, L_2, L_3 form an *asymtotic triangle* iff $L_i \cdot L_j \in C$, and $L_i \cdot L_j \neq L_i \cdot L_k$ for $i \neq j \neq k \neq i$, and $i, j, k = 1, 2, 3$.

One can easily verify that L_1, L_2, L_3 form an asymptotic triangle iff there exist lines M_1, M_2, M_3 such that L_i, M_i are orthogonal for $i = 1, 2, 3$ and there is no line orthogonal to any two lines from the set $\{L_1, L_2, L_3\}$ (Fig. 1).

3. The incidence orthogonal structure $(\mathscr{I}, \mathscr{L}_2, \mathscr{L}_0)$. In contradiction to the projective points and lines the elements of the sets $\mathscr{I}, \mathscr{L}_2, \mathscr{L}_0$ will be called *H-points* (*hyperbolic points*), *H-secant lines* and *H-exterior lines* respectively.

The H-line L *meets* H-line M iff $L \neq M$ and there exists H-point p lying on L and M.

Let us fix that H-secant lines contain the odd or infinite (resp. even) number of H-points.

The H-point p is an *H-pole* of the H-exterior line L iff each H-secant line (resp. H-exterior line) passing through p meets L.

The H-exterior line L is an *H-polar* of H-point p iff each H-line passing through p and meeting L is H-secant (resp. H-exterior) line.

The H-line M is *orthogonal to* the *H-exterior line* L iff M passes through the H-pole of L.

The H-exterior line M is *orthogonal to* the *H-secant line* L iff M is the H-polar of some H-point on L.

The H-secant line M is *orthogonal to* the *H-secant line* L iff M is disjoint with each H-exterior line orthogonal to L and M meets L (resp. M is disjoint with L).

Two H-lines are *parallel* iff there is no H-line orthogonal to each of them. If two H-lines are parallel then they are different H-secant lines.

Three H-lines L_1, L_2, L_3 form an *H-asymptotic triangle* iff L_i and L_j are parallel for $i \neq j$, $i, j = 1, 2, 3$ and there exist H-lines M_1, M_2, M_3 such that L_i and M_i are orthogonal for $i = 1, 2, 3$, and M_i, L_j are parallel for $i \neq j$, $i, j = 1, 2, 3$.

The *pencil of I-type* is the set of all H-lines passing through the same H-point.

The *pencil of II-type* is the set of all H-lines such that every two H-lines are parallel and no three H-lines form an H-asymptotic triangle.

The *pencil of III-type* is the set of all H-lines orthogonal to the same H-secant line.

Three H-points are *collinear* iff there exists H-line passing through each of these points.

The bijection of $\mathscr{I}$ preserving collinearity of H-points is called the *H-collineation*. One can easily verify that each H-collineation transforms the

H-secant lines onto H-secant lines, H-exterior lines onto H-exterior lines and preserves orthogonality of H-lines. Hence H-collineation transforms the pencil of i-type onto pencil of i-type for $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$.

COROLLARY. *Each H-collineation may be extend to the collineation of Π, i.e. the group of the H-collineations is the set of all restrictions of the collineations of Π preserving C* [1].

Since on real hyperbolic plane $\mathscr{L}_0 = \varnothing$ then the above characterization of orthogonality is impossible there (the other but similar inner characterization is given in [3]).

PROBLEM. Find all fields satisfying Lemma 1 or Lemma 2, and all the incidence structures $(\mathscr{I}, \mathscr{L}_2, \mathscr{L}_0)$ where the above characterization of orthogonality is possible.

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