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REGULAR SOLUTIONS OF SOME FUNCTIONAL EQUATIONS IN THE INDETERMINATE CASE

Abstract. The paper deals with the existence and uniqueness of regular solutions of the equation $\varphi(x) = h(x, \varphi(f(x)))$. Also in the indeterminate case the existence of solutions of $\varphi(f(x)) = g(x, \varphi(x))$ is studied.

In the present paper we shall consider the following functional equations

(1) $$\varphi(x) = h(x, \varphi[f(x)]),$$

(2) $$\varphi[f(x)] = g[x, \varphi(x)],$$

where $\varphi : I = [0, a) \to R$, $0 < a \leqslant \infty$ is an unknown function.

The phrase *regular solution* used in the title will have the following meaning: "a solution which is continuous in the whole interval I and possesses a right-side derivative at the point zero".

The problem of regular solutions of linear functional equations is contained in [1] and [2]. The theory of continuous solutions of equations (1) and (2) has been developed in [4], [5], [6], [7], [8].

§ 1. Let I be an interval $[0, a)$, $0 < a \leqslant \infty$ and let Ω be a neighbourhood of $(0, 0) \in R^2$. Assume that the given functions f, g and h fulfil the following conditions.

(i) The function $f : I \to R$ is continuous, strictly increasing, there exists $f'(0+) \neq 0$ and $0 < f(x) < x$ in $I \setminus \{0\}$.

(ii) The function $h : \Omega \to R$ is continuous, there exist $c > 0$, $d > 0$ and a continuous fuction $\gamma : [0, c) \subset I \to R$ such that

(3) $$|h(x, y_1) - h(x, y_2)| \leqslant \gamma(x)\, |y_1 - y_2| \text{ in } U \cap \Omega,$$

where $U : 0 \leqslant x < c$, $|y| < d$. Moreover, there exist A and B such that

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$$h(x, y) = Ax + By + R(x, y), \tag{4}$$

where

$$R(x, y) = o\left(\sqrt{x^2 + y^2}\right), \quad (x, y) \to (0, 0). \tag{5}$$

For every $x \in I$ we denote

$$\Omega_x := \{y : (x, y) \in \Omega\} \tag{6}$$

and

$$A_x := \{h(x, y) : y \in \Omega_x\}.$$

(iii) For every $x \in I$, Ω_x is an open interval and $A_{f(x)} = \Omega_x$.

(iv) The function $g : \Omega \to R$ is continuous for every $x \in I$. For every fixed $x \in I$ the function g as a function of y is invertible. There exist $c > 0$, $d > 0$ and a continuous function $\gamma : [0, c) \subset I \to R$ such that

$$|g(x, y_1) - g(x, y_2)| \leqslant \gamma(x)\, |y_1 - y_2| \text{ in } U \cap \Omega, \tag{7}$$

where $U : 0 \leqslant x < c$, $|y| < d$ and γ has a positive bound in $[0, c)$. Moreover, there exist A and $B \neq 0$ such that

$$g(x, y) = Ax + By + R(x, y), \tag{8}$$

where

$$R(x, y) = o\left(\sqrt{x^2 + y^2}\right), \quad (x, y) \to (0, 0).$$

For every $x \in I$ we denote

$$V_x := \{g(x, y) : y \in \Omega_x\},$$

where Ω_x is given by (6).

(v) For every $x \in I$, Ω_x is an open interval and $V_x = \Omega_{f(x)}$.

The indeterminate case

$$f'(0)\, \gamma(0) = 1 \tag{9}$$

for equation (1) and

$$\frac{f'(0)}{\gamma(0)} = 1 \tag{10}$$

for equation (2) will be considered in this paper.

§ 2. Let us consider equation (1) and let $\varphi(x) = x\, \psi(x)$. Then equation (1) is of the form

$$\psi(x) = H(x, \varphi[f(x)]), \tag{11}$$

where

$$H(x, z) := \begin{cases} \dfrac{1}{x} h(x, f(x)\, z), & \text{for } x \neq 0 \\ A + B f'(0)\, z & \text{for } x = 0 \end{cases} \tag{12}$$

is defined in the set

$$\Omega^1 = \{(x, z) : (x, f(x)\, z) \in \Omega\}.$$

For an arbitrary $x \in I$ we denote

$$\Omega^1_x := \{z : (x, z) \in \Omega^1\}$$

and

$$A^1_x := \{H(x, z) : z \in \Omega^1_x\}.$$

LEMMA 1. *Assume hypotheses* (i), (ii), (iii) *and condition* (9). *Then the function H given by the formula* (12) *fulfils the following conditions:*

(a) *H is continuous in a neighbourhood of the point $(0, \eta)$, where η is arbitrary constant.*

(b) *For every η, there exist a $\delta \in (0, c]$ and a $d_1 > 0$ such that*

$$|H(x, z_1) - H(x, z_2)| \leqslant \gamma_1(x)\, |z_1 - z_2| \text{ for } x \in [0, \delta),\ |z - \eta| < d_1, \tag{13}$$

where

$$\gamma_1(x) = \begin{cases} \dfrac{f(x)}{x}\, \gamma(x), & \text{for } x \in (0, \delta) \\ 1, & \text{for } x = 0 \end{cases} \tag{14}$$

is a continuous function.

(c) *For every $x \in I$, Ω^1_x is an open interval and $A^1_{f(x)} \subset \Omega^1_x$.*

P r o o f. Ad (a). It suffices to show, that the function H is continuous at the point $(0, \eta)$. We have

$$\lim_{\substack{x \to 0+ \\ z \to \eta}} H(x, z) = \lim_{\substack{x \to 0+ \\ z \to \eta}} \frac{1}{x}\, h[x, f(x)\, z] = \lim_{\substack{x \to 0+ \\ z \to \eta}} \left[A + B\, \frac{f(x)}{x}\, z + \frac{1}{x}\, R(x, f(x)\, z)\right].$$

From (5) we obtain

$$\lim_{\substack{x \to 0+ \\ z \to \eta}} \frac{R(x, f(x)\, z)}{\sqrt{x^2 + [f(x)\, z]^2}} = 0$$

whereas (i) implies

$$\left| \frac{R(x, f(x)\, z)}{\sqrt{x^2 + [f(x)\, z]^2}} \right| \geqslant \left| \frac{R(x, f(x)\, z)}{x\sqrt{1 + z^2}} \right|.$$

Therefore,

$$\lim_{\substack{x \to 0+ \\ z \to \eta}} H(x, z) = A + B\, f'(0)\, \eta = H(0, \eta),$$

which finishes our proof in case (a).

Ad (b). The function f is continuous and $f(0) = 0$; thus there exist a $\delta \in (0, c]$ and a $d_1 > 0$ such that for every $x \in (0, \delta)$ and $|z - \eta| < d_1$ the inequality $|f(x)\, z| < d$ holds. Therefore, the inequality

$$|H(x, z_1) - H(x, z_2)| = \left| \frac{1}{x}\, [h(x, f(x)\, z_1) - h(x, f(x)\, z_2)] \right| \leqslant$$

$$\leqslant \frac{f(x)}{x}\, \gamma(x)\, |z_1 - z_2| \text{ for } x \neq 0$$

holds and

$$|H(0, z_1)-H(0, z_2)| = |B|\, f'(0)\, |z_1-z_2|.$$

From (3) we have

$$|B| = |h_y(0, 0)| \leqslant \gamma(0).$$

Consequently,

$$|B|\, f'(0) \leqslant \gamma(0)\, f'(0) = 1.$$

Condition (13) is fulfilled with the function γ_1 given by formula (14). We have also

$$\lim_{x\to 0} \frac{f(x)}{x}\,\gamma(x) = f'(0)\gamma(0) = 1.$$

This implies that γ_1 is continuous in $[0, c)$.

Ad (c). We may assume that Ω is of the form

$$\Omega : \begin{cases} 0 \leqslant x < a \\ \alpha_1(x) < y < \alpha_2(x), \end{cases}$$

where $\alpha_1(x) < 0$ and $\alpha_2(x) > 0$ for $x \in [0, a)$. It is easily seen that

$$\Omega^1 : \begin{cases} 0 \leqslant x < a \\ \dfrac{\alpha_1(x)}{f(x)} < z < \dfrac{\alpha_2(x)}{f(x)} \text{ for } x \neq 0 \\ z \text{ is arbitrary for } x = 0. \end{cases}$$

In particular, we have $\Omega_0^1 = (-\infty, +\infty)$ whence $\Lambda^1_{f(0)} \subset \Omega^1_0$. For $x \neq 0$ we have

$$\Lambda^1_{f(x)} = \{v : v = H\,[f(x), z],\ z \in \Omega^1_{f(x)}\} =$$

$$= \left\{v : v = \frac{1}{f(x)}\, h\,[f(x),\ f^2(x)\, z],\ z \in \left(\frac{\alpha_1[f(x)]}{f^2(x)},\ \frac{\alpha_2[f(x)]}{f^2(x)}\right)\right\} =$$

$$= \{v : f(x)\, v = h[f(x), y],\ y \in (\alpha_1[f(x)], \alpha_2[f(x)]) = \Omega_{f(x)}\}.$$

From (iii) we obtain

$$\Lambda_{f(x)} = \{h\,[f(x), y] : y \in \Omega_{f(x)}\} \subset \Omega_x$$

whence

$$\frac{\alpha_1(x)}{f(x)} < v < \frac{\alpha_2(x)}{f(x)}.$$

Therefore

$$\Lambda^1_{f(x)} \subset \left(\frac{\alpha_1(x)}{f(x)},\ \frac{\alpha_2(x)}{f(x)}\right) = \Omega^1_x.$$

The proof of Lemma 1 is complete.

From Lemma 1 and condition (11) we obtain.

LEMMA 2. *We assume* (i), (ii), (iii)). *If ψ is a continuous solution of equation* (11) *in I, then $\varphi(x) = x\,\psi(x)$ is a regular solution of equation* (1) *in I and such that $\varphi(0)=0$. If φ is a regular solution of equation* (1) *in I and such that $\varphi(0)=0$ and $\varphi'(0)=\eta$, then the function*

$$\psi(x)=\begin{cases}\dfrac{\varphi(x)}{x} & \text{for } x\neq 0\\ \varphi'(0) & \text{for } x=0\end{cases}$$

yields a continuous solution of equation (11) *in I such that $\psi(0)=\eta$.*

The uniqueness of regular solutions depends essentially on the behaviour of the sequence

$$\Gamma_n(x):=\prod_{i=0}^{n-1}\gamma_1[f^i(x)], \tag{15}$$

where γ_1 is defined by (14).

THEOREM 1. *We assume* (i), (ii) (iii) *and* (9). *Let η be an arbitrary constant. If there exist an $M>0$ and a $\delta_1\in(0,\delta]$ (where δ is the constant from* Lemma 1) *such that*

$$\Gamma_n(x)\leqslant M,\ n=0,1,\dots \text{ for } x\in[0,\delta_1),$$

then equation (1) *has at most one regular solution in I fulfilling conditions $\varphi(0)=0$, $\varphi'(0)=\eta$.*

P r o o f. From our hypotheses and from Lemma 1 it follows that the assumptions of Theorem 1 from [4] are fulfilled. Thus equation (11) has at most one continuous solution in I fulfilling the condition $\psi(0)=\eta$. Now, our assertion results from Lemma 2.

REMARK 1. If equation (11) has a continuous solution, then

$$H(0,\eta)=\eta. \tag{16}$$

The definition of function H implies that equation (16) assumes the form

$$A+B\,f'(0)\,\eta=\eta.$$

Let

$$\mathcal{H}(x):=|H(x,\eta)-\eta|=\begin{cases}\left|\dfrac{1}{x}h[x,f(x)\,\eta]-\eta\right| & \text{for } x\neq 0\\ 0 & \text{for } x=0.\end{cases} \tag{17}$$

THEOREM 2. *We assume* (i), (ii), (iii) *and condition* (9). *If (for a fixed η fulfilling equation* (16)) *there exists a $\delta_2>0$ such that*

$$\sum_{n=0}^{\infty}\Gamma_n(x)\,\mathcal{H}\,[f^n(x)] \tag{18}$$

is uniformly convergent in $[0,\delta_2)$ then equation (1) *has a regular solution in I fulfilling conditions $\varphi(0)=0$, $\varphi'(0)=\eta$.*

P r o o f. From the hypotheses of our theorem and from Lemma 1 it follows that the assumptions of Theorem 3 from [4] concerning equation (11) are satisfied. Consequently, equation (11) possesses a continuous solution in I fulfilling the condition $\psi(0) = \eta$. Now, our assertion results from Lemma 2.

§ 3. If we put $\varphi(x) = x\,\psi(x)$ in equation (2), then we come to

$$\psi[f(x)] = K[x, \psi(x)], \tag{19}$$

where the function

$$K(x, z) = \begin{cases} \dfrac{1}{f(x)}\, g(x, x\,z) & \text{for } x \neq 0 \\ \dfrac{A}{f'(0)} + \dfrac{B}{f'(0)}\, z & \text{for } x = 0 \end{cases} \tag{20}$$

is defined in the region

$$\Omega^2 = \{(x, z) : (x, xz) \in \Omega\}.$$

For any fixed $x \in I$ we put

$$\Omega^2_x = \{y : (x, y) \in \Omega^2\}$$

and

$$V^2_x = \{K(x, z) : z \in \Omega^2_x\}.$$

LEMMA 3. *Suppose that assumptions* (i), (iv), (v) *and condition* (10) *are satisfied. Then the function* $K(x, z)$ *fulfils the following conditions:*

(a) $K(x, z)$ *is continuous in a neighbourhood of each point* $(0, \eta)$, *where* η *is an arbitrary constant.*

(b) *For every real* η *there exist constants* $\delta \in (0, c]$ *and* $d_1 > 0$ *such that*

$$|K(x, z_1) - K(x, z_2)| \leqslant \gamma_1(x)\, |z_1 - z_2| \quad \text{for } 0 \leqslant x < \delta,\ |z - \eta| < d_1, \tag{21}$$

where

$$\gamma_1(x) = \begin{cases} \dfrac{x}{f(x)}\, \gamma(x) & \text{for } x \in (0, \delta) \\ 1 & \text{for } x = 0 \end{cases} \tag{22}$$

is a continuous function.

(c) *For every* $x \in I$ *the set* Ω^2_x *is an open interval and* $V^2_x = \Omega^2_{f(x)}$.

(d) *For any fixed* $x \in I$ *the function* $K(x, z)$ *is invertible with respect to* z.

P r o o f. Ad (a). In order to prove that $K(x, z)$ is continuous it suffices to show its continuity at a point $(0, \eta)$;

$$\lim_{\substack{x \to 0+ \\ z \to \eta}} K(x, z) = \lim_{\substack{x \to 0+ \\ z \to \eta}} \frac{1}{f(x)}\, g(x, xz) =$$

$$= \lim_{\substack{x \to 0+ \\ z \to \eta}} \left[A\frac{x}{f(x)} + B\,\frac{x}{f(x)}\, z + \frac{1}{f(x)}\, R(x, xz) \right].$$

Hypothesis (8) implies

$$\lim_{\substack{x \to 0+ \\ z \to \eta}} \frac{R(x, xz)}{x\sqrt{1+z^2}} = 0,$$

whence

$$\lim_{\substack{x \to 0+ \\ z \to \eta}} \frac{1}{f(x)} R(x, xz) = \lim_{\substack{x \to 0+ \\ z \to \eta}} \frac{R(x, xz)}{x} \cdot \frac{x}{f(x)} = 0.$$

Consequently

$$\lim_{\substack{x \to 0+ \\ z \to \eta}} K(x, z) = \frac{A}{f'(0)} + \frac{B}{f'(0)} \eta = K(0, \eta),$$

which finishes the proof of (a).

Ad (b). Evidently, for every real η one can find a $\delta \in (0, c]$ and a positive d_1 such that the inequality $|xz| < d$ is satisfied whenever $x \in (0, \delta)$ and $|z-\eta| < d_1$. Applying condition (7) we get

$$|K(x, z_1)-K(x, z_2)| = \left|\frac{1}{f(x)} [g(x, xz_1)-g(x,xz_2)]\right| \leqslant \frac{x}{f(x)} \gamma(x) |z_1-z_2| \text{ for } x \neq 0.$$

For $x = 0$ we have

$$|K(0, z_1)-K(0, z_2)| = \frac{|B|}{f'(0)} |z_1-z_2|.$$

Condition (7) implies also that

$$|B| = |g_y(0, 0)| \leqslant \gamma(0).$$

Since $0 < f(x) < x$, we have $f'(0) = \lim_{x \to 0} \frac{f(x)}{x} > 0$, because $f'(0) \neq 0$ by assumption. Consequently, on account of (10) we have

$$\frac{|B|}{f'(0)} \leqslant \frac{\gamma(0)}{f'(0)} = 1.$$

This proves that condition (21) is satisfied with $\gamma_1(x)$ defined by (22). Since

$$\lim_{x \to 0+} \frac{x}{f(x)} \gamma(x) = \frac{\gamma(0)}{f'(0)} = 1,$$

the function $\gamma_1(x)$ is continuous in $[0, \delta)$ which completes the proof of condition (b).

Ad (c). We may assume that the domain Ω has the form

$$\Omega : \begin{cases} 0 \leqslant x < a \\ \alpha_1(x) < y < \alpha_2(x), \end{cases}$$

where $\alpha_1(x) < 0$ and $\alpha_2(x) > 0$. It is easy to check that, in such a case, the region Ω^2 is of the form

$$\Omega^2 : \begin{cases} 0 \leqslant x < a \\ \dfrac{a_1(x)}{x} < z < \dfrac{a_2(x)}{x} & \text{for } x \neq 0, \\ z \text{ is arbitrary} & \text{for } x = 0. \end{cases}$$

Since $f(0) = 0$ we have $\Omega^2_{f(0)} = \Omega^2_0 = (-\infty, \infty)$. On the other hand

$$K(0, z) = \frac{A}{f'(0)} + \frac{B}{f'(0)} z, \text{ whence } V^2_0 = (-\infty, \infty)$$

i.e. $V^2_0 = \Omega^2_{f(0)}$. For $x \neq 0$ we have

$$V^2_x = \left\{ v : v = \frac{1}{f(x)} g(x, xz), z \in \left(\frac{a_1(x)}{x}, \frac{a_2(x)}{x} \right) \right\} =$$
$$= \{ v : f(x)\, v = g(x, y),\ y \in \Omega_x \}.$$

Assumption (v) implies

$$\{ f(x)\, v : f(x)\, v = g(x, y),\ y \in \Omega_x \} = (a_1[f(x)],\ a_2[f(x)])$$

whence

$$V^2_x = \left(\frac{a_1[f(x)]}{f(x)}, \frac{a_2[f(x)]}{f(x)} \right) = \Omega^2_{f(x)},$$

which proves our assertion (c).

Ad (d). Since $K(0, z) = \dfrac{A}{f'(0)} + \dfrac{B}{f'(0)} z$ and $B \neq 0$ by assumption, function $K(0, z)$ is invertible. For $x \neq 0$ the function $K(x, z)$ is a one-to-one mapping with respect to z because $g(x, y)$ is invertible as a function of the second variable (with an arbitrarily fixed $x \in I$). This proves (d).

The following lemma is a simple consequence of equation (19) and Lemma 3.

LEMMA 4. *Assume* (i), (iv) *and* (v). *If* ψ *is a continuous solution of equation* (19) *then the function* $\varphi(x) = x\, \psi(x)$ *is a regular solution of equation* (2) *fulfilling the condition* $\varphi(0) = 0$. *If* φ *is a regular solution of equation* (2) *in* I *such that* $\varphi(0) = 0$ *and* $\varphi'(0) = \eta$, *then the function*

$$\psi(x) = \begin{cases} \dfrac{\varphi(x)}{x} & \textit{for } x \neq 0 \\ \varphi'(0) & \textit{for } x = 0 \end{cases}$$

is a continuous solution of (19) *fulfilling the condition* $\psi(0) = \eta$.

Let $\Gamma_n(x)$ be defined by formula (13) where $\gamma_1(x)$ is given by (22).

THEOREM 3. *Assume* (i), (iv), (v) *and condition* (10). *Moreover, suppose that there exists an interval* $J \subset I$ *such that* $\Gamma_n(x)$ *tends to zero uniformly on* J. *Then a regular solution of equation (2) fulfilling the conditions* $\varphi(0) = 0$ *and* $\varphi'(0) = \eta$, *depends on an arbitrary function.*

P r o o f. On account of our assumptions and by means of Lemma 3 we infer that the assumptions of Theorem 6 from [4] concerning equation (19) are satisfied. Thus a continuous solution ψ of (19) fulfilling the condition $\psi(0) = \eta$ (if such a solution exists) depends on an arbitrary function. Now, Lemma 4 completes our proof.

Condition

$$K(0, \eta) = \eta \tag{23}$$

is necessary for equation (19) to have a continuous solution with $\psi(0) = \eta$.

REMARK 2. The definition of $K(x, z)$ implies that equation (23) has the form

$$\frac{A}{f'(0)} + \frac{B}{f'(0)}\eta = \eta.$$

Put

$$\mathcal{K}(x) := |K(x, \eta) - \eta| = \begin{cases} \left| \frac{1}{f(x)} g(x, x\eta) - \eta \right| & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases} \tag{24}$$

where η is a solution of (23) and

$$H_n(x) := \sum_{i=0}^{n-2} \frac{\mathcal{K}[f^i(x)]}{\Gamma_{i+1}(x)} \Gamma_n(x), \; n = 2, 3, \ldots \tag{25}$$

THEOREM 4. *Assume* (i), (iv), (v) *and condition* (10). *If, for a fixed η, there exists a point $x_0 \in I\setminus\{0\}$ such that both $\Gamma_n(x)$ and $H_n(x)$ tend to zero uniformly on $[f(x_0), x_0]$, then equation* (2) *has a regular solution φ in I fulfilling the conditions $\varphi(0) = 0$ and $\varphi'(0) = \eta$, depending on an arbitrary function.*

Proof. The assumptions of our theorem and Lemma 3 imply that the hypotheses of Theorem 7 from [4] concerning equation (19) are satisfied. Consequently, equation (19) has a continuous solution ψ in I fulfilling the condition $\psi(0) = \eta$ and depending on an arbitrary function. Now our assertion results from Lemma 4.

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