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ONE-PARAMETER GENERALIZATION OF DUAL-HYPERBOLIC JACOBSTHAL NUMBERS

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Abstract. In this paper, we introduce one-parameter generalization of dual-hyperbolic Jacobsthal numbers – dual-hyperbolic r-Jacobsthal numbers. We present some properties of them, among others the Binet formula, Catalan, Cassini, and d'Ocagne identities. Moreover, we give the generating function and summation formula for these numbers. The presented results are a generalization of the results for the dual-hyperbolic Jacobsthal numbers.

1. Introduction

A hyperbolic number is defined as $h = x + yj$, where $x, y \in \mathbb{R}$ and j is a unipotent (hyperbolic) imaginary unit such that $j^2 = 1$ and $j \neq \pm 1$. Hence, the set of hyperbolic numbers is defined as

$$\mathbb{H} = \{h\colon h = x + yj,\ x, y \in \mathbb{R},\ j^2 = 1\}.$$

The hyperbolic imaginary unit j was introduced by Cockle, [5, 6, 7, 8]. Hy-

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perbolic numbers are well studied in the literature, see [12]. The set of dual numbers is defined in the following way

$$\mathbb{D} = \{d\colon d = u + v\varepsilon,\ u, v \in \mathbb{R},\ \varepsilon^2 = 0,\ \varepsilon \neq 0\}.$$

Dual numbers were introduced by Clifford, see [4]. Dual-hyperbolic numbers are known generalizations of hyperbolic and dual numbers.

The set of dual-hyperbolic numbers, denoted by $\mathbb{DH}$, is defined as follows

$$\mathbb{DH} = \{w\colon w = a_1 + a_2 j + (a_3 + a_4 j)\varepsilon = a_1 + a_2 j + a_3\varepsilon + a_4 j\varepsilon\colon a_1, a_2, a_3, a_4 \in \mathbb{R}\},$$

where the base elements $(1, j, \varepsilon, j\varepsilon)$ of dual-hyperbolic numbers correspond to the following commutative multiplications

$$j^2 = 1,\ \varepsilon^2 = (j\varepsilon)^2 = 0,\ \varepsilon(j\varepsilon) = (j\varepsilon)\varepsilon = 0,\ j(j\varepsilon) = (j\varepsilon)j = \varepsilon. \tag{1.1}$$

Let $w_1 = a_1 + a_2 j + a_3\varepsilon + a_4 j\varepsilon$, $w_2 = b_1 + b_2 j + b_3\varepsilon + b_4 j\varepsilon$ be any two dual-hyperbolic numbers. Then the equality, addition, subtraction and multiplication by scalar are defined in the natural way:

$$\begin{gathered}
w_1 = w_2 \quad \text{only if} \quad a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4,\\
w_1 + w_2 = a_1 + b_1 + (a_2 + b_2)j + (a_3 + b_3)\varepsilon + (a_4 + b_4)j\varepsilon,\\
w_1 - w_2 = a_1 - b_1 + (a_2 - b_2)j + (a_3 - b_3)\varepsilon + (a_4 - b_4)j\varepsilon,\\
\text{for} \quad k \in \mathbb{R},\ kw_1 = ka_1 + ka_2 j + ka_3\varepsilon + ka_4 j\varepsilon.
\end{gathered}$$

By (1.1) we get

$$\begin{aligned}
w_1 \cdot w_2 = a_1 b_1 + a_2 b_2 + (a_1 b_2 + a_2 b_1)j + (a_1 b_3 + a_2 b_4 + a_3 b_1 + a_4 b_2)\varepsilon\\
+ (a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1)j\varepsilon.
\end{aligned} \tag{1.2}$$

The dual-hyperbolic numbers form a commutative ring, a real vector space, and an algebra. Many interesting properties of the dual-hyperbolic numbers are given in [1].

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = 0$, $F_1 = 1$. There are many numbers defined by the linear recurrence relations and they are also named as numbers of the Fibonacci type, for example, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers.

The Jacobsthal sequence $\{J_n\}$ is defined by the recurrence

$$J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2$$

with the initial conditions $J_0 = 0$, $J_1 = 1$. The first ten terms of the sequence are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171$. The Binet formula of this sequence has the form

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{for } n \geq 0.$$

Many authors studied some generalizations of the recurrence of the Jacobsthal sequence, see [9, 10]. In [2], a one-parameter generalization of the Jacobsthal numbers was introduced. We recall this generalization.

Let $n \geq 0$, $r \geq 0$ be integers. The nth r-Jacobsthal number $J(r, n)$ is defined as follows

$$J(r,n) = 2^r J(r, n-1) + (2^r + 4^r)J(r, n-2) \quad \text{for } n \geq 2 \tag{1.3}$$

with $J(r, 0) = 1$, $J(r, 1) = 1 + 2^{r+1}$. For $r = 0$, we have $J(0, n) = J_{n+2}$. By (1.3) we obtain

$$\begin{aligned}
J(r,0) &= 1,\\
J(r,1) &= 2 \cdot 2^r + 1,\\
J(r,2) &= 3 \cdot 4^r + 2 \cdot 2^r,\\
J(r,3) &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r,\\
J(r,4) &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r,\\
J(r,5) &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r.
\end{aligned} \tag{1.4}$$

In [2], it was proved that the r-Jacobsthal numbers can be used for the counting of independent sets of special classes of graphs. We will recall some properties of the r-Jacobsthal numbers.

Theorem 1.1 (Binet formula [2]). *Let $n \geq 0$, $r \geq 0$ be integers. Then the nth r-Jacobsthal number is given by*

$$J(r,n) = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} {\lambda_1}^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} {\lambda_2}^n,$$

where

$$\lambda_1 = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \lambda_2 = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Theorem 1.2 ([2]). *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$\sum_{l=0}^{n-1} J(r,l) = \frac{J(r,n) + (2^r + 4^r)J(r,n-1) - 2 - 2^r}{4^r + 2^{r+1} - 1}.$$

Theorem 1.3 (Cassini identity [2]). *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$J(r,n+1)J(r,n-1) - J^2(r,n) = (-1)^n(2^r+1)^2(2^r+4^r)^{n-1}.$$

Theorem 1.4 (Convolution identity [2]). *Let $n \geq 1$, $m \geq 2$, $r \geq 0$ be integers. Then*

$$J(r,m+n) = 2^r J(r,m-1)J(r,n) + (4^r + 8^r)J(r,m-2)J(r,n-1).$$

Numbers of the Fibonacci type and their generalizations appear in many subjects of mathematics. These numbers have applications also in the theory of complex numbers and the theory of quaternions, see [16]. In [11], Horadam defined the Fibonacci and Lucas quaternions. In [14, 15], the authors studied Jacobsthal and Pell quaternions. In [3], the dual-hyperbolic Fibonacci and Lucas numbers were introduced. In [13], the authors investigated dual-hyperbolic Jacobsthal and Jacobsthal-Lucas numbers. Based on these ideas, we define and study dual-hyperbolic r-Jacobsthal numbers.

2. Dual-hyperbolic r-Jacobsthal numbers

For integers $n \geq 0$, $r \geq 0$, we define the nth dual-hyperbolic r-Jacobsthal number $\mathbb{DH}J(r,n)$ by the following relation

$$\mathbb{DH}J(r,n) = J(r,n) + J(r,n+1)j + J(r,n+2)\varepsilon + J(r,n+3)j\varepsilon, \tag{2.1}$$

where $J(r,n)$ is the nth r-Jacobsthal number.

By (1.4) and (2.1), we obtain

$$\begin{aligned}
\mathbb{DH}J(r,0) &= 1 + (2^{r+1}+1)j + (3\cdot 4^r + 2^{r+1})\varepsilon + (5\cdot 8^r + 5\cdot 4^r + 2^r)j\varepsilon,\\
\mathbb{DH}J(r,1) &= 2^{r+1} + 1 + (3\cdot 4^r + 2^{r+1})j + (5\cdot 8^r + 5\cdot 4^r + 2^r)\varepsilon\\
&\quad + (8\cdot 16^r + 10\cdot 8^r + 3\cdot 4^r)j\varepsilon, \qquad (2.2)\\
\mathbb{DH}J(r,2) &= 3\cdot 4^r + 2^{r+1} + (5\cdot 8^r + 5\cdot 4^r + 2^r)j\\
&\quad + (8\cdot 16^r + 10\cdot 8^r + 3\cdot 4^r)\varepsilon + (13\cdot 32^r + 20\cdot 16^r + 9\cdot 8^r + 4^r)j\varepsilon.
\end{aligned}$$

Using the fact that $J(0,n) = J_{n+2}$, we obtain $\mathbb{DH}J(0,n) = \mathbb{DH}J_{n+2}$, where $\mathbb{DH}J_n$ is the nth dual-hyperbolic Jacobsthal number introduced in [13].

In the set $\mathbb{C}$, the complex conjugate of $x + yi$ is $(x + yi)^* = x - yi$. In $\mathbb{DH}$, for a dual-hyperbolic number $w = a_1 + a_2 j + a_3\varepsilon + a_4 j\varepsilon$, there are five different conjugations (see [1, 3]).

The hyperbolic conjugation of $\mathbb{DH}J(r,n)$ has the form

$$\mathbb{DH}J(r,n)^{*1} = J(r,n) - J(r,n+1)j + J(r,n+2)\varepsilon - J(r,n+3)j\varepsilon,$$

the dual conjugation of $\mathbb{DH}J(r,n)$ has the form

$$\mathbb{DH}J(r,n)^{*2} = J(r,n) + J(r,n+1)j - J(r,n+2)\varepsilon - J(r,n+3)j\varepsilon,$$

the coupled conjugation of $\mathbb{DH}J(r,n)$ has the form

$$\mathbb{DH}J(r,n)^{*3} = J(r,n) - J(r,n+1)j - J(r,n+2)\varepsilon + J(r,n+3)j\varepsilon,$$

the dual-hyperbolic conjugation of $\mathbb{DH}J(r,n)$ has the form

$$\mathbb{DH}J(r,n)^{*4} = J(r,n) - J(r,n+1)j - 1 + \frac{J(r,n+2) + J(r,n+3)j}{J(r,n) + J(r,n+1)j}\varepsilon$$

and the anti-dual conjugation of $\mathbb{DH}J(r,n)$ has the form

$$\mathbb{DH}J(r,n)^{*5} = J(r,n+2) + J(r,n+3)j - J(r,n)\varepsilon + J(r,n+1)j\varepsilon.$$

In $\mathbb{C}$, the modulus of $x + yi$ is $|x + yi| = \sqrt{(x + yi)(x + yi)^*} = \sqrt{x^2 + y^2}$. In $\mathbb{DH}$, there are five different moduli. Now we give the values of the squares of these modules:

$$\begin{aligned}
\left(N_{\mathbb{DH}J(r,n)^{*1}}\right)^2 &= \mathbb{DH}J(r,n) \cdot \mathbb{DH}J(r,n)^{*1} \\
&= J^2(r,n) - J^2(r,n+1) + 2[J(r,n)J(r,n+2) \\
&\quad - J(r,n+1)J(r,n+3)]\varepsilon, \\
\left(N_{\mathbb{DH}J(r,n)^{*2}}\right)^2 &= \mathbb{DH}J(r,n) \cdot \mathbb{DH}J(r,n)^{*2} \\
&= J^2(r,n) + J^2(r,n+1) + 2J(r,n)J(r,n+1)j, \\
\left(N_{\mathbb{DH}J(r,n)^{*3}}\right)^2 &= \mathbb{DH}J(r,n) \cdot \mathbb{DH}J(r,n)^{*3} \\
&= J^2(r,n) - J^2(r,n+1) + 2[J(r,n)J(r,n+3) \\
&\quad - J(r,n+1)J(r,n+2)]j\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\left(N_{\mathbb{DH}J(r,n)^{*4}}\right)^2 &= \mathbb{DH}J(r,n)\cdot\mathbb{DH}J(r,n)^{*4}\\
&= J^2(r,n) - J^2(r,n+1) - \mathbb{DH}J(r,n)\\
&\quad + [J(r,n)J(r,n+2) - J(r,n+1)J(r,n+3)\\
&\quad + J(r,n+2)]\varepsilon + [J(r,n)J(r,n+3)\\
&\quad - J(r,n+1)J(r,n+2) + J(r,n+3)]j\varepsilon,\\
\left(N_{\mathbb{DH}J(r,n)^{*5}}\right)^2 &= \mathbb{DH}J(r,n)\cdot\mathbb{DH}J(r,n)^{*5}\\
&= J(r,n)J(r,n+2) + J(r,n+1)J(r,n+3)\\
&\quad + [J(r,n)J(r,n+3) + J(r,n+1)J(r,n+2)]j\\
&\quad + [-J^2(r,n) + J^2(r,n+1) + J^2(r,n+2) + J^2(r,n+3)]\varepsilon\\
&\quad + 2J(r,n+2)J(r,n+3)j\varepsilon.
\end{aligned}$$

By the definition of the dual-hyperbolic r-Jacobsthal numbers, we get the following recurrence relations.

Proposition 2.1. *Let $n \geq 2$, $r \geq 0$ be integers. Then*

$$\mathbb{DH}J(r,n) = 2^r\mathbb{DH}J(r,n-1) + (2^r + 4^r)\mathbb{DH}J(r,n-2)$$

where $\mathbb{DH}J(r,0)$, $\mathbb{DH}J(r,1)$ are given by (2.2).

Theorem 2.2. *Let $n \geq 0$, $r \geq 0$ be integers. Then*

$$\begin{aligned}
&\mathbb{DH}J(r,n) - j\mathbb{DH}J(r,n+1) - \varepsilon\mathbb{DH}J(r,n+2) - j\varepsilon\mathbb{DH}J(r,n+3)\\
&\qquad\qquad = J(r,n) - J(r,n+2) - 2(J(r,n+3)j + J(r,n+4))\varepsilon.
\end{aligned}$$

Proof. By simple calculations, we get

$$\begin{aligned}
&\mathbb{DH}J(r,n) - j\mathbb{DH}J(r,n+1) - \varepsilon\mathbb{DH}J(r,n+2) - j\varepsilon\mathbb{DH}J(r,n+3)\\
&\quad = J(r,n) + J(r,n+1)j + J(r,n+2)\varepsilon + J(r,n+3)j\varepsilon\\
&\qquad - j\left(J(r,n+1) + J(r,n+2)j + J(r,n+3)\varepsilon + J(r,n+4)j\varepsilon\right)\\
&\qquad - \varepsilon\left(J(r,n+2) + J(r,n+3)j + J(r,n+4)\varepsilon + J(r,n+5)j\varepsilon\right)\\
&\qquad - j\varepsilon\left(J(r,n+3) + J(r,n+4)j + J(r,n+5)\varepsilon + J(r,n+6)j\varepsilon\right)
\end{aligned}$$

$$
\begin{aligned}
&= J(r,n) + J(r,n+1)j + J(r,n+2)\varepsilon + J(r,n+3)j\varepsilon \\
&\quad - J(r,n+1)j - J(r,n+2) - J(r,n+3)j\varepsilon - J(r,n+4)\varepsilon \\
&\quad - J(r,n+2)\varepsilon - J(r,n+3)j\varepsilon - J(r,n+3)j\varepsilon - J(r,n+4)\varepsilon \\
&= J(r,n) - J(r,n+2) - 2(J(r,n+3)j + J(r,n+4))\varepsilon. \qquad \square
\end{aligned}
$$

Now, we will present the Binet formula for the dual-hyperbolic r-Jacobsthal numbers.

Theorem 2.3 (Binet formula for dual-hyperbolic r-Jacobsthal numbers). *Let $n \geq 0$, $r \geq 0$ be integers. Then*

$$
\mathbb{DH}J(r,n) = C_1 \underline{\alpha} \alpha^n + C_2 \underline{\beta} \beta^n, \tag{2.3}
$$

where

$$
\alpha = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \beta = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r},
$$

$$
\underline{\alpha} = 1 + \alpha j + \alpha^2 \varepsilon + \alpha^3 j\varepsilon, \quad \underline{\beta} = 1 + \beta j + \beta^2 \varepsilon + \beta^3 j\varepsilon,
$$

$$
C_1 = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}, \quad C_2 = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}.
$$

Proof. By the Binet formula for the r-Jacobsthal numbers, we obtain

$$
\begin{aligned}
\mathbb{DH}J(r,n) &= J(r,n) + J(r,n+1)j + J(r,n+2)\varepsilon + J(r,n+3)j\varepsilon \\
&= C_1\alpha^n + C_2\beta^n + (C_1\alpha^{n+1} + C_2\beta^{n+1})j \\
&\quad + (C_1\alpha^{n+2} + C_2\beta^{n+2})\varepsilon + (C_1\alpha^{n+3} + C_2\beta^{n+3})j\varepsilon \\
&= C_1\alpha^n \left(1 + \alpha j + \alpha^2\varepsilon + \alpha^3 j\varepsilon\right) + C_2\beta^n \left(1 + \beta j + \beta^2\varepsilon + \beta^3 j\varepsilon\right) \\
&= C_1 \underline{\alpha} \alpha^n + C_2 \underline{\beta} \beta^n. \qquad \square
\end{aligned}
$$

Corollary 2.4. (*Binet formula for dual-hyperbolic Jacobsthal numbers*) *Let $n \geq 0$ be an integer. Then*

$$
\mathbb{DH}J_n = \frac{1}{3}\left[2^n(1 + 2j + 4\varepsilon + 8j\varepsilon) - (-1)^n(1 - j + \varepsilon - j\varepsilon)\right].
$$

Proof. By formula (2.3), for $r = 0$, we have $C_1 = \frac{4}{3}$, $C_2 = -\frac{1}{3}$, $\alpha = 2$, $\beta = -1$, and

$$\begin{aligned}\mathbb{DH}J(0,n) &= \frac{4}{3}\cdot 2^n(1+2j+4\varepsilon+8j\varepsilon) - \frac{1}{3}(-1)^n(1-j+\varepsilon-j\varepsilon)\\ &= \frac{1}{3}\cdot 2^{n+2}(1+2j+4\varepsilon+8j\varepsilon) - \frac{1}{3}(-1)^{n+2}(1-j+\varepsilon-j\varepsilon)\\ &= \mathbb{DH}J_{n+2}. \qquad \square\end{aligned}$$

The Binet formula (2.3) can be used for proving some identities for the dual-hyperbolic r-Jacobsthal numbers. We will need the following lemma.

Lemma 2.5. *Let*

$$\alpha = 2^{r-1} + \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \quad \beta = 2^{r-1} - \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r},$$
$$\underline{\alpha} = 1 + \alpha j + \alpha^2\varepsilon + \alpha^3 j\varepsilon, \quad \underline{\beta} = 1 + \beta j + \beta^2\varepsilon + \beta^3 j\varepsilon.$$

Then

$$\begin{aligned}\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha} &= 1 - 4^r - 2^r + 2^r j + (4^r + 2^{r+1} - 5\cdot 8^r - 3\cdot 16^r)\varepsilon\\ &\quad + (3\cdot 8^r + 2\cdot 4^r)j\varepsilon.\end{aligned} \tag{2.4}$$

Proof. By simple calculations, we get

$$\begin{aligned}\underline{\alpha}\underline{\beta} &= 1 + \alpha\beta + (\alpha+\beta)j + (\alpha^2+\beta^2)(1+\alpha\beta)\varepsilon\\ &\quad + (\alpha^3+\beta^3+\alpha\beta(\alpha+\beta))j\varepsilon.\end{aligned}$$

Using the equalities

$$\begin{aligned}\alpha + \beta &= 2^r,\\ \alpha - \beta &= \sqrt{4\cdot 2^r + 5\cdot 4^r},\\ \alpha\beta &= -(4^r + 2^r),\\ \alpha^2 + \beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta = 3\cdot 4^r + 2^{r+1},\\ \alpha^3 + \beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = 4\cdot 8^r + 3\cdot 4^r,\end{aligned}$$

we get the result. $\square$

3. Some identities for the dual-hyperbolic r-Jacobsthal numbers

In this section, we give some identities such as Catalan, Cassini, and d'Ocagne identities for the dual-hyperbolic r-Jacobsthal numbers.

Theorem 3.1 (Catalan identity for dual-hyperbolic r-Jacobsthal numbers). *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq m$. Then*

$$(\mathbb{DH}J(r,n))^2 - \mathbb{DH}J(r,n-m) \cdot \mathbb{DH}J(r,n+m)$$
$$= -\frac{(-4^r - 2^r)^n (1+2^r)^2}{4 \cdot 2^r + 5 \cdot 4^r} \underline{\alpha\beta} \left(2 - \left(\frac{\alpha}{\beta}\right)^m - \left(\frac{\beta}{\alpha}\right)^m\right),$$

where $\underline{\alpha\beta}$ is given by (2.4).

Proof. By formula (2.3), we get

$$\begin{aligned}
&(\mathbb{DH}J(r,n))^2 - \mathbb{DH}J(r,n-m) \cdot \mathbb{DH}J(r,n+m)\\
&= (C_1\underline{\alpha}\alpha^n + C_2\underline{\beta}\beta^n)(C_1\underline{\alpha}\alpha^n + C_2\underline{\beta}\beta^n)\\
&\quad - (C_1\underline{\alpha}\alpha^{n-m} + C_2\underline{\beta}\beta^{n-m})(C_1\underline{\alpha}\alpha^{n+m} + C_2\underline{\beta}\beta^{n+m})\\
&= 2C_1C_2\underline{\alpha\beta}(\alpha\beta)^n - C_1C_2\underline{\alpha\beta}\alpha^{n+m}\beta^{n-m} - C_1C_2\underline{\alpha\beta}\alpha^{n-m}\beta^{n+m}\\
&= C_1C_2\underline{\alpha\beta}(\alpha\beta)^n \left(2 - \left(\frac{\alpha}{\beta}\right)^m - \left(\frac{\beta}{\alpha}\right)^m\right).
\end{aligned}$$

Using the fact that $\alpha\beta = -(4^r + 2^r)$, we obtain

$$(\mathbb{DH}J(r,n))^2 - \mathbb{DH}J(r,n-m) \cdot \mathbb{DH}J(r,n+m)$$
$$= C_1C_2(-4^r - 2^r)^n \underline{\alpha\beta} \left(2 - (\frac{\alpha}{\beta})^m - (\frac{\beta}{\alpha})^m\right),$$

where

$$C_1C_2 = -\frac{(1+2^r)^2}{4 \cdot 2^r + 5 \cdot 4^r}$$

and $\underline{\alpha\beta}$ is given by (2.4). $\square$

For $m = 1$, we obtain Cassini identity for the dual-hyperbolic r-Jacobsthal numbers.

COROLLARY 3.2 (Cassini identity for dual-hyperbolic r-Jacobsthal numbers). *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$(\mathbb{DH}J(r,n))^2 - \mathbb{DH}J(r,n-1) \cdot \mathbb{DH}J(r,n+1) = (-4^r - 2^r)^{n-1}(1+2^r)^2 \underline{\alpha\beta}.$$

PROOF. By simple calculations we have

$$\begin{aligned}(\mathbb{DH}J(r,n))^2 &- \mathbb{DH}J(r,n-1) \cdot \mathbb{DH}J(r,n+1) \\ &= -C_1C_2\underline{\alpha\beta}(\alpha\beta)^{n-1}(\alpha-\beta)^2.\end{aligned}$$

Using the fact that $\alpha - \beta = \sqrt{4 \cdot 2^r + 5 \cdot 4^r}$, $\alpha\beta = -(4^r + 2^r)$, and $C_1C_2 = -\frac{(1+2^r)^2}{4 \cdot 2^r + 5 \cdot 4^r}$, we get the result. $\square$

In particular, by Theorem 3.1, we obtain the following formulas for the dual-hyperbolic Jacobsthal numbers.

COROLLARY 3.3 (Catalan identity for dual-hyperbolic Jacobsthal numbers). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$(\mathbb{DH}J_n)^2 - \mathbb{DH}J_{n-m} \cdot \mathbb{DH}J_{n+m} = \frac{4}{9}(-2)^{n-m}\left((-2)^m - 1\right)^2(-1 + j - 5\varepsilon + 5j\varepsilon).$$

COROLLARY 3.4 (Cassini identity for dual-hyperbolic Jacobsthal numbers). *Let $n \geq 1$ be an integer. Then*

$$(\mathbb{DH}J_n)^2 - \mathbb{DH}J_{n-1} \cdot \mathbb{DH}J_{n+1} = 4(-2)^{n-1}(-1 + j - 5\varepsilon + 5j\varepsilon).$$

THEOREM 3.5 (d'Ocagne identity for dual-hyperbolic r-Jacobsthal numbers). *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq m$. Then*

$$\begin{aligned}\mathbb{DH}J(r,n) &\cdot \mathbb{DH}J(r,m+1) - \mathbb{DH}J(r,n+1) \cdot \mathbb{DH}J(r,m) \\ &= \frac{(1+2^r)^2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}{4 \cdot 2^r + 5 \cdot 4^r}(-4^r - 2^r)^m \underline{\alpha\beta}\left(\alpha^{n-m} - \beta^{n-m}\right),\end{aligned}$$

where $\underline{\alpha\beta}$ is given by (2.4).

Proof. By formula (2.3), we get

$$\begin{aligned}
&\mathbb{DH}J(r,n)\cdot\mathbb{DH}J(r,m+1)-\mathbb{DH}J(r,n+1)\cdot\mathbb{DH}J(r,m)\\
&\quad=(C_1\underline{\alpha}\alpha^n+C_2\underline{\beta}\beta^n)(C_1\underline{\alpha}\alpha^{m+1}+C_2\underline{\beta}\beta^{m+1})\\
&\qquad-(C_1\underline{\alpha}\alpha^{n+1}+C_2\underline{\beta}\beta^{n+1})(C_1\underline{\alpha}\alpha^m+C_2\underline{\beta}\beta^m)\\
&\quad=C_1C_2\underline{\alpha}\underline{\beta}\left(\alpha^{m+1}\beta^n+\alpha^n\beta^{m+1}-\alpha^m\beta^{n+1}-\alpha^{n+1}\beta^m\right)\\
&\quad=C_1C_2\underline{\alpha}\underline{\beta}(\alpha\beta)^m(\alpha\beta^{n-m}+\alpha^{n-m}\beta-\beta^{n-m+1}-\alpha^{n-m+1})\\
&\quad=C_1C_2(\beta-\alpha)(\alpha\beta)^m\underline{\alpha}\underline{\beta}\left(\alpha^{n-m}-\beta^{n-m}\right)\\
&\quad=\frac{(1+2^r)^2\sqrt{4\cdot 2^r+5\cdot 4^r}}{4\cdot 2^r+5\cdot 4^r}(-4^r-2^r)^m\underline{\alpha}\underline{\beta}\left(\alpha^{n-m}-\beta^{n-m}\right). \qquad \square
\end{aligned}$$

Corollary 3.6 (d'Ocagne identity for dual-hyperbolic Jacobsthal numbers). *Let $n\geq 0$, $m\geq 0$ be integers such that $n\geq m$. Then*

$$\begin{aligned}
&\mathbb{DH}J_n\cdot\mathbb{DH}J_{m+1}-\mathbb{DH}J_{n+1}\cdot\mathbb{DH}J_m\\
&\qquad=\frac{4}{3}(-2)^m\left(2^{n-m}-(-1)^{n-m}\right)(-1+j-5\varepsilon+5j\varepsilon).
\end{aligned}$$

The next theorem gives the convolution identity for the dual-hyperbolic r-Jacobsthal numbers.

Theorem 3.7. *Let $m\geq 2, n\geq 1$, $r\geq 0$ be integers. Then*

$$\begin{aligned}
2\mathbb{DH}J(r,m+n)&=2^r\mathbb{DH}J(r,m-1)\cdot\mathbb{DH}J(r,n)\\
&\quad+(4^r+8^r)\mathbb{DH}J(r,m-2)\cdot\mathbb{DH}J(r,n-1)+J(r,m+n)\\
&\quad-J(r,m+n+2)-2J(r,m+n+4)\varepsilon-2J(r,m+n+3)j\varepsilon.
\end{aligned}$$

Proof. By simple calculations, using (1.2), we have

$$\begin{aligned}
&\mathbb{DH}J(r,m-1)\cdot\mathbb{DH}J(r,n)\\
&\quad=J(r,m-1)J(r,n)+J(r,m-1)J(r,n+1)j\\
&\qquad+J(r,m-1)J(r,n+2)\varepsilon+J(r,m-1)J(r,n+3)j\varepsilon\\
&\qquad+J(r,m)J(r,n)j+J(r,m)J(r,n+1)+J(r,m)J(r,n+2)j\varepsilon\\
&\qquad+J(r,m)J(r,n+3)\varepsilon+J(r,m+1)J(r,n)\varepsilon+J(r,m+1)J(r,n+1)j\varepsilon\\
&\qquad+J(r,m+2)J(r,n)j\varepsilon+J(r,m+2)J(r,n+1)\varepsilon.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbb{DH}J(r,m-2)\cdot\mathbb{DH}J(r,n-1)\\
&\quad= J(r,m-2)J(r,n-1)+J(r,m-2)J(r,n)j\\
&\quad+J(r,m-2)J(r,n+1)\varepsilon+J(r,m-2)J(r,n+2)j\varepsilon\\
&\quad+J(r,m-1)J(r,n-1)j+J(r,m-1)J(r,n)\\
&\quad+J(r,m-1)J(r,n+1)j\varepsilon+J(r,m-1)J(r,n+2)\varepsilon\\
&\quad+J(r,m)J(r,n-1)\varepsilon+J(r,m)J(r,n)j\varepsilon\\
&\quad+J(r,m+1)J(r,n-1)j\varepsilon+J(r,m+1)J(r,n)\varepsilon.
\end{aligned}$$

Let $A=2^r, B=4^r+8^r$. Then

$$\begin{aligned}
&A\cdot\mathbb{DH}J(r,m-1)\cdot\mathbb{DH}J(r,n)+B\cdot\mathbb{DH}J(r,m-2)\cdot\mathbb{DH}J(r,n-1)\\
&\quad= A\,J(r,m-1)J(r,n)+B\,J(r,m-2)J(r,n-1)\\
&\quad+[A\,J(r,m-1)J(r,n+1)+B\,J(r,m-2)J(r,n)\\
&\quad+A\,J(r,m)J(r,n)+B\,J(r,m-1)J(r,n-1)]j\\
&\quad+[A\,J(r,m-1)J(r,n+2)+B\,J(r,m-2)J(r,n+1)\\
&\quad+A\,J(r,m+1)J(r,n)+B\,J(r,m)J(r,n-1)]\varepsilon\\
&\quad+[A\,J(r,m-1)J(r,n+3)+B\,J(r,m-2)J(r,n+2)\\
&\quad+A\,J(r,m)J(r,n+2)+B\,J(r,m-1)J(r,n+1)]j\varepsilon\\
&\quad+A\,J(r,m)J(r,n+1)+B\,J(r,m-1)J(r,n)\\
&\quad+[A\,J(r,m)J(r,n+3)+B\,J(r,m-1)J(r,n+2)\\
&\quad+A\,J(r,m+2)J(r,n+1)+B\,J(r,m+1)J(r,n)]\varepsilon\\
&\quad+[A\,J(r,m+1)J(r,n+1)+B\,J(r,m)J(r,n)\\
&\quad+A\,J(r,m+2)J(r,n)+B\,J(r,m+1)J(r,n-1)]j\varepsilon.
\end{aligned}$$

Using Theorem 1.4, we get

$$\begin{aligned}
&2^r\mathbb{DH}J(r,m-1)\cdot\mathbb{DH}J(r,n)+(4^r+8^r)\mathbb{DH}J(r,m-2)\cdot\mathbb{DH}J(r,n-1)\\
&\quad= J(r,m+n)+2[J(r,m+n+1)j+J(r,m+n+2)\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&+ J(r,m+n+3)j\varepsilon] + J(r,m+n+2)\\
&+ 2J(r,m+n+4)\varepsilon + 2J(r,m+n+3)j\varepsilon\\
&= 2\mathbb{DH}J(r,m+n) - J(r,m+n) + J(r,m+n+2)\\
&+ 2J(r,m+n+4)\varepsilon + 2J(r,m+n+3)j\varepsilon.
\end{aligned}
$$

Hence we get the result. □

Now, we give a summation formula for the dual-hyperbolic r-Jacobsthal numbers.

THEOREM 3.8. *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$
\begin{aligned}
\sum_{l=0}^{n} \mathbb{DH}J(r,l) = \frac{1}{4^r + 2^{r+1} - 1}&[\mathbb{DH}J(r,n+1) + (2^r + 4^r)\mathbb{DH}J(r,n)\\
&- (1 + j + \varepsilon + j\varepsilon)(2 + 2^r)]\\
&- j - (2 + 2^{r+1})\varepsilon - (2^{r+2} + 3 \cdot 4^r + 2)j\varepsilon.
\end{aligned}
$$

PROOF. Using Theorem 1.2, we get

$$
\begin{aligned}
\sum_{l=0}^{n} \mathbb{DH}J(r,l) &= \sum_{l=0}^{n}(J(r,l) + J(r,l+1)j + J(r,l+2)\varepsilon + J(r,l+3)j\varepsilon)\\
&= \sum_{l=0}^{n} J(r,l) + \sum_{l=0}^{n} J(r,l+1)j\\
&\quad + \sum_{l=0}^{n} J(r,l+2)\varepsilon + \sum_{l=0}^{n} J(r,l+3)j\varepsilon\\
&= \frac{1}{4^r + 2^{r+1} - 1}[J(r,n+1) + (2^r + 4^r)J(r,n) - 2 - 2^r\\
&\quad + (J(r,n+2) + (2^r + 4^r)J(r,n+1) - 2 - 2^r)j\\
&\quad + (J(r,n+3) + (2^r + 4^r)J(r,n+2) - 2 - 2^r)\varepsilon\\
&\quad + (J(r,n+4) + (2^r + 4^r)J(r,n+3) - 2 - 2^r)j\varepsilon] - J(r,0)j\\
&\quad - (J(r,0) + J(r,1))\varepsilon - (J(r,0) + J(r,1) + J(r,2))j\varepsilon.
\end{aligned}
$$

By simple calculations, we obtain

$$\sum_{l=0}^{n}\mathbb{DH}J(r,l) = \frac{1}{4^r+2^{r+1}-1}[J(r,n+1)+J(r,n+2)j$$

$$+ J(r,n+3)\varepsilon + J(r,n+4)j\varepsilon$$

$$+ (2^r+4^r)(J(r,n)+J(r,n+1)j+J(r,n+2)\varepsilon+J(r,n+3)j\varepsilon)$$

$$- (2+2^r)(1+j+\varepsilon+j\varepsilon)] - j - (2^{r+1}+2)\varepsilon - (2^{r+2}+3\cdot 4^r+2)j\varepsilon$$

$$= \frac{\mathbb{DH}J(r,n+1)+(2^r+4^r)\mathbb{DH}J(r,n)-(1+j+\varepsilon+j\varepsilon)(2+2^r)}{4^r+2^{r+1}-1}$$

$$- j - (2+2^{r+1})\varepsilon - (2^{r+2}+3\cdot 4^r+2)j\varepsilon. \qquad \square$$

At the end, we give the generating function for the dual-hyperbolic r-Jacobsthal numbers. We recall the result for the r-Jacobsthal sequence.

THEOREM 3.9 ([2]). *The generating function of the sequence of r-Jacobsthal numbers has the following form*

$$f(t) = \frac{1+(1+2^r)t}{1-2^rt-(2^r+4^r)t^2}.$$

THEOREM 3.10. *The generating function for the dual-hyperbolic r-Jacobsthal numbers has the following form*

$$g(t) = \frac{\mathbb{DH}J(r,0)+(\mathbb{DH}J(r,1)-2^r\mathbb{DH}J(r,0))t}{1-2^rt-(2^r+4^r)t^2}.$$

PROOF. Let

$$g(t) = \mathbb{DH}J(r,0)+\mathbb{DH}J(r,1)t+\mathbb{DH}J(r,2)t^2+\cdots+\mathbb{DH}J(r,n)t^n+\cdots$$

be the generating function of the dual-hyperbolic r-Jacobsthal numbers. Then

$$2^rtg(t) = 2^r\mathbb{DH}J(r,0)t+2^r\mathbb{DH}J(r,1)t^2+2^r\mathbb{DH}J(r,2)t^3$$

$$+\cdots+2^r\mathbb{DH}J(r,n-1)t^n+\cdots,$$

$$(2^r+4^r)t^2g(t) = (2^r+4^r)\mathbb{DH}J(r,0)t^2+(2^r+4^r)\mathbb{DH}J(r,1)t^3$$

$$+(2^r+4^r)\mathbb{DH}J(r,2)t^4+\cdots$$

$$+(2^r+4^r)\mathbb{DH}J(r,n-2)t^n+\cdots.$$

By Proposition 2.1, we get

$$\begin{aligned}
g(t) - 2^r t g(t) - (2^r + 4^r)t^2 g(t) \\
&= \mathbb{DH}J(r,0) + (\mathbb{DH}J(r,1) - 2^r \mathbb{DH}J(r,0))t \\
&\quad + (\mathbb{DH}J(r,2) - 2^r \mathbb{DH}J(r,1) - (2^r + 4^r)\mathbb{DH}J(r,0))t^2 + \cdots \\
&= \mathbb{DH}J(r,0) + (\mathbb{DH}J(r,1) - 2^r \mathbb{DH}J(r,0))t.
\end{aligned}$$

Thus

$$g(t) = \frac{\mathbb{DH}J(r,0) + (\mathbb{DH}J(r,1) - 2^r \mathbb{DH}J(r,0))t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

Using equality (2.2), we obtain

$$\begin{aligned}
\mathbb{DH}J(r,0) &= 1 + (2^{r+1} + 1)j + (3 \cdot 4^r + 2^{r+1})\varepsilon \\
&\quad + (5 \cdot 8^r + 5 \cdot 4^r + 2^r)j\varepsilon, \\
\mathbb{DH}J(r,1) - 2^r \mathbb{DH}J(r,0) &= 2^r + 1 + (4^r + 2^r)j + (2 \cdot 8^r + 3 \cdot 4^r + 2^r)\varepsilon \\
&\quad + (3 \cdot 16^r + 5 \cdot 8^r + 2 \cdot 4^r)j\varepsilon. \qquad \square
\end{aligned}$$

4. Compliance with ethical standards

Conflict of Interest: The authors declare that they have no conflict of interest.

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