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IDENTITIES ARISING FROM BINOMIAL-LIKE FORMULAS INVOLVING DIVISORS OF NUMBERS

Karol Gryszka

Abstract. In this article, we derive a great number of identities involving the ω function counting distinct prime divisors of a given number n. These identities also include Pochhammer symbols, Fibonacci and Lucas numbers and many more.

1. Introduction and preliminaries

In the recent articles [2, 8] several formulas involving prime omega function and binomial-like expansion were derived. The present paper extends the results obtained there by introducing many identities arising from the formulas included in [2]. These identities combine Fibonacci, Lucas, Stirling and Lah numbers. Such identities already exist in the literature and they usually involve binomial coefficients [1, 6]. For example, in [1] one can find the following formula

$$\sum_{m=0}^{v} s(v,m) \sum_{j=0}^{n} \binom{n}{j} m^j \ell_{n-j}(v) L_n = \begin{cases} 0, & n \neq v, \\ n! L_v, & n = v, \end{cases}$$

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where

$$\frac{w^v}{\prod_{j=0}^{v-1}(e^w - j)} = \sum_{n=0}^{\infty} \ell_n(v)\frac{w^n}{n!}$$

and $s(v,m)$ are the Stirling numbers of the first kind. In [6] we find

$$\sum_{v_1+\cdots+v_k=n} \frac{L_{v_1}\cdots L_{v_k}}{v_1\cdots v_k} = \frac{k!}{n!}\sum_{j=k}^{n}(-1)^{j-k}(n-j)!\binom{n}{j}\binom{j}{n-j}s(j,k).$$

Note that the literature contains many identities involving the ω function, see for instance [3]. We can also find many results related to the growth rate of the function [3, 4]. In this article we focus on the identities directly related to the values of ω in combination with other known integer sequences, which one side is usually or similar to

$$\sum_{d|n} f(x,\omega(d)),$$

and the other side is in some way a closed form of the latter. Then, using particular substitutions for x we obtain many peculiar identities (which are presented in Section 3), including the identities involving Lucas, Fibonacci and Stirling or Lah numbers (which are derived in Section 4). It seems that such a topic is covered by very few articles (such as [9]) and we hope that this article will encourage further research.

Throughout the article, let $n \geq 2$ be an integer with canonical factorization

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

where p_i's are prime numbers and a_i's are positive integers. We define the prime omega function $\omega\colon \mathbb{N}_1 \to \mathbb{N}$ counting the number of distinct prime factors [4], that is,

$$\omega(n) := \begin{cases} k, & n = \prod_{i=1}^{k} p_i^{a_i}, \\ 0, & n = 1. \end{cases}$$

We shall often abuse the notation and write $n = \prod_{i=1}^{\omega(n)} p_i^{a_i}$ (so that the value of $\omega(n)$ is not explicitly stated as equal to some k).

Recall the notion of Pochhammer symbols and their basic properties. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we set

$$x^{\underline{n}} = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1}(x-k), & n > 0, \end{cases} \qquad x^{\overline{n}} = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1}(x+k), & n > 0. \end{cases}$$

They are also called rising and falling factorials, respectively. If x is a positive integer, they satisfy the following simple identities

$$x^{\overline{n}} = \frac{(x+n-1)!}{(x-1)!}, \quad x^{\underline{n}} = \frac{x!}{(x-n)!}.$$

If $\mathbb{R}[X_1, \dots, X_k]$ is a ring of polynomials in k variables over the field of real numbers, then elementary symmetric polynomials $S_m(X_1, \dots, X_k)$ are defined as the sums of all distinct products of m variables, that is:

$$\begin{aligned} S_0(X_1, \dots, X_k) &= 1, \\ S_1(X_1, \dots, X_k) &= X_1 + \dots + X_k, \\ &\vdots \\ S_{k-1}(X_1, \dots, X_k) &= \sum_{1 \le i_1 < i_2 < \dots < i_{k-1} \le k} \prod_{j=1}^{k-1} X_{i_j}, \\ S_k(X_1, \dots, X_k) &= X_1 \cdots X_k. \end{aligned}$$

See [5] for further details concerning symmetric polynomials.

From now on, whenever we use symmetric polynomials, we assume the number $n = \prod_{i=1}^{\omega(n)} p_i^{a_i}$ is given and we let $S_i(a)$ denote the value of S_i on the vector $a = (a_1, \dots, a_{\omega(n)})$.

Recall the following theorem [2, 8].

THEOREM 1.1. *If $n > 0$ is a square-free number, then*

$$(x+y)^{\overline{\omega(n)}} = \sum_{d|n} x^{\overline{\omega(n)-\omega(d)}} y^{\overline{\omega(d)}}. \tag{1.1}$$

The above can also be expressed for general n, but in a slightly less convenient way [2]:

THEOREM 1.2. *For any positive integer n we have*

$$(x+y)^{\omega(n)} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(n)-\omega(d)} y^{\omega(d)}. \tag{1.2}$$

Formulas (1.1) and (1.2) have their special cases obtained after substitution $x \mapsto 1$ and $y \mapsto x$. These are, respectively,

$$(1+x)^{\omega(n)} = \sum_{d|n} x^{\omega(d)}, \tag{1.3}$$

$$(1+x)^{\omega(n)} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)}. \tag{1.4}$$

REMARK 1.3. Let us note that the difference between the above formulas is in the coefficient $\frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)}$, which actually differentiate square-free case from the non square-free case. This is true for all future results in this article.

In Theorem 1.1, we assume that n is a square-free number. The formula (1.1) is a special case of the following formula (see also [2]).

THEOREM 1.4. *For arbitrary integer $n > 0$ and any $x, y \in \mathbb{R}$ we have*

$$\prod_{i=1}^{\omega(n)} (x + a_i y) = \sum_{d|n} x^{\omega(n)-\omega(d)} y^{\omega(d)}. \tag{1.5}$$

2. Some simple observations and identities

We now present some simple observations based on formulas (1.1)–(1.5). Note that in the article we use the convention $0^0 = 1$.

EXAMPLE 2.1. Recall from [2] that if we substitute $x = y = 1$ and $x = 1$, $y = -1$ in (1.5), then we obtain

$$\prod_{i=1}^{\omega(n)} (1 + a_i) = \sum_{d|n} 1, \qquad \prod_{i=1}^{\omega(n)} (1 - a_i) = \sum_{d|n} (-1)^{\omega(d)}.$$

The left identity is a classic formula for the number of divisors of arbitrary number n. For the right identity, if there is a prime number p such that $v_p(n) = 1$ (the p-adic valuation of n, i.e. the largest such k that $p^k|n$), then the right-hand-side sum equals 0.

Another observation comes from substitution $x = 0$, $y = 1$ in (1.5). In this case we obtain the formula for the product of exponents:

$$\prod_{i=1}^{\omega(n)} a_i = \sum_{\substack{d|n,\\ \omega(d)=\omega(n)}} 0^{\omega(d)-\omega(n)} 1^{\omega(d)} = \sum_{\substack{d|n,\\ \omega(d)=\omega(n)}} 1 = \#\{d|n \mid \omega(d) = \omega(n)\}.$$

As a result, the identity (1.5) can be seen as a generalization of the formula for the number of divisors of n.

Example 2.2. If we substitute $x = 1$ and $x = -1$ in (1.3), we obtain, respectively, for any square-free number n,

$$\sum_{d|n} 1 = 2^{\omega(n)}, \quad \sum_{d|n} (-1)^{\omega(d)} = 0.$$

The second formula is equivalent to

$$\sum_{d|n \,\wedge\, 2|\omega(d)} 1 = \sum_{d|n \,\wedge\, 2\nmid\omega(d)} 1. \tag{2.1}$$

The case of square-free number is special, since it is clear that $\#\{d|n \mid \omega(d) = m\} = \binom{\omega(n)}{m}$. Therefore, formula (2.1) resembles the classic binomial equality

$$\sum_{2|i} \binom{\omega(n)}{i} = \sum_{2\nmid i} \binom{\omega(n)}{i}.$$

Example 2.3. The formula (1.5) is not symmetric, i.e., the equality

$$\sum_{d|n} x^{\omega(n)-\omega(d)} y^{\omega(d)} = \sum_{d|n} y^{\omega(n)-\omega(d)} x^{\omega(d)}$$

does not hold in general. This is not the case for the usual binomial formula, where

$$\sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} x^i$$

holds due to properties of binomial coefficients. On the other hand, the formula (1.2) is symmetric.

Before we present the next identity, we need the following helpful lemma describing the property of symmetric polynomials.

Lemma 2.4. *For any positive integer N and non-negative integers $a_1, \dots, a_N$ the following holds:*

$$\sum_{i=1}^{N}\left[a_i\prod_{j\neq i}(1+a_j)\right]=\sum_{i=1}^{N} iS_i(a_1,\dots,a_N).$$

Proof. If $N=1$, the left-hand-side and right-hand-side both equal a_1. The case $N=2$ gives

$$a_1(1+a_2)+a_2(1+a_1)=(a_1+a_2)+2a_1a_2$$

and shows the identity in the case $N=2$.

The general case follows from induction on N. Let $N\geq 2$ and suppose the lemma holds for N. Recall that

$$a_{N+1}S_{i-1}(a_1,\dots,a_N)+S_i(a_1,\dots,a_N)=S_i(a_1,\dots,a_{N+1}).$$

Then,

$$\begin{aligned}
\sum_{i=1}^{N+1}\left[a_i\prod_{j\neq i}(1+a_j)\right] &= (1+a_{N+1})\sum_{i=1}^{N}\left[a_i\prod_{j\neq i, j\leq N}(1+a_j)\right]+a_{N+1}\prod_{j=0}^{N}(1+a_j)\\
&= (1+a_{N+1})\sum_{i=1}^{N} iS_i(a_1,\dots,a_N)+a_{N+1}\sum_{i=0}^{N}S_i(a_1,\dots,a_N)\\
&= (1+a_{N+1})\sum_{i=1}^{N-1} iS_i(a_1,\dots,a_N)+(1+a_{N+1})NS_N(a_1,\dots,a_N)\\
&\quad + a_{N+1}\sum_{i=1}^{N-1}S_i(a_1,\dots,a_N)+a_{N+1}+a_{N+1}S_N(a_1,\dots,a_N)\\
&= \sum_{i=1}^{N-1}(i+1)a_{N+1}S_i(a_1,\dots,a_N)+\sum_{i=1}^{N-1} iS_i(a_1,\dots,a_N)\\
&\quad + a_{N+1}+NS_N(a_1,\dots,a_N)+(N+1)a_{N+1}S_N(a_1,\dots,a_N)
\end{aligned}$$

$$= \sum_{i=1}^{N} i\big[a_{N+1}S_{i-1}(a_1,\dots,a_N) + S_i(a_1,\dots,a_N)\big]$$

$$+ (N+1)S_{N+1}(a_1,\dots,a_{N+1})$$

$$= \sum_{i=1}^{N} iS_i(a_1,\dots,a_{N+1}) + (N+1)S_{N+1}(a_1,\dots,a_{N+1}).$$

This completes the induction. □

EXAMPLE 2.5. If we substitute $x \mapsto 1$ and $y \mapsto x$ in (1.5), then

$$\prod_{i=1}^{\omega(n)} (1 + a_i x) = \sum_{d|n} x^{\omega(d)}.$$

Let us now compute the logarithmic derivative of the above equation and multiply the result by x:

$$\sum_{i=1}^{\omega(n)} \frac{a_i x}{1 + a_i x} = \frac{\sum\limits_{d|n} \omega(d) x^{\omega(d)}}{\sum\limits_{d|n} x^{\omega(d)}}.$$

Using (1.5) we find that

$$\sum_{d|n} \omega(d) x^{\omega(d)} = \sum_{i=1}^{\omega(n)} \left[\frac{a_i x}{(1 + a_i x)} \prod_{j=1}^{\omega(n)} (1 + a_j x) \right] = \sum_{i=1}^{\omega(n)} \left[a_i x \prod_{j \neq i} (1 + a_j x) \right].$$

Plugging $x = 1$ and using Lemma 2.4 we obtain the identity

$$\sum_{d|n} \omega(d) = \sum_{i=1}^{\omega(n)} iS_i(a).$$

We will derive more general formula for the above equality in the next section.

3. Main identities

3.1. A generalization

We shall generalize the identity obtained in Example 2.5. First, recall the following formula obtained in [2].

LEMMA 3.1. *Suppose* $n = \prod_{i=1}^{\omega(n)} p_i^{a_i}$ *is a canonical factorization of* n. *Then*

$$\prod_{i=1}^{\omega(n)} (x + a_i y) = \sum_{i=0}^{\omega(n)} S_i(a) x^{\omega(n)-i} y^i.$$

In particular,

$$\sum_{d|n} x^{\omega(d)} = \sum_{i=0}^{\omega(n)} S_i(a) x^i.$$

THEOREM 3.2. *For any positive integer* n *and* $m \in \mathbb{Z}$ *we have*

$$\sum_{d|n, d>1} \omega(d)^m x^{\omega(d)} = \sum_{k=1}^{\omega(n)} k^m S_k(a) x^k. \tag{3.1}$$

In particular,

$$\sum_{d|n, d>1} \omega(d)^m = \sum_{k=1}^{\omega(n)} k^m S_k(a).$$

PROOF. The case $m = 0$ is covered by Lemma 3.1.

For the remaining cases the formula is obtained by induction on m.

First, we obtain the formula for $m > 0$. The case $m = 1$ with $x = 1$ is derived in Example 2.5 (it can also be derived from the induction step presented below). Let $m \geq 0$ and suppose that (3.1) holds for $m \geq 0$. Differentiating (3.1) with respect to x we obtain

$$\sum_{d|n, d>1} \omega(d)^{m+1} x^{\omega(d)-1} = \sum_{k=1}^{\omega(n)} k^{m+1} S_k(a) x^{k-1}.$$

Multiplying by x we get the formula (3.1) with $m \mapsto m + 1$.

Now, we obtain the formula for $m < 0$. Suppose (3.1) holds for some $m \leq 0$. Then diving that by x we obtain

$$\sum_{d|n, d>1} \omega(d)^m x^{\omega(d)-1} = \sum_{k=1}^{\omega(n)} k^m S_k(a) x^{k-1}.$$

Thus, integrating with respect to x we find

$$\sum_{d|n, d>1} \omega(d)^{m-1} x^{\omega(d)} = \sum_{k=1}^{\omega(n)} k^{m-1} S_k(a) x^k + C.$$

Substitution $x = 0$ yields $C = 0$ and proves the formula (3.1) with $m \mapsto m-1$, completing the induction. □

EXAMPLE 3.3. Let $n = 360 = 2^3 \cdot 3^2 \cdot 5$. Then

$$S_0(3,2,1) = 1,\ S_1(3,2,1) = 6,\ S_2(3,2,1) = 11,\ S_3(3,2,1) = 6.$$

If $m = 3$, then

$$\begin{aligned}\sum_{d|n, d>1} \omega(d)^3 &= 1^3 \cdot S_1(3,2,1) + 2^3 \cdot S_2(3,2,1) + 3^3 \cdot S_3(3,2,1) \\ &= 1 \cdot 6 + 8 \cdot 11 + 27 \cdot 6 = 256.\end{aligned}$$

Similarly, if $m = -2$, then

$$\sum_{d|n, d>1} \frac{1}{\omega(d)^2} = \frac{S_1(3,2,1)}{1^2} + \frac{S_2(3,2,1)}{2^2} + \frac{S_3(3,2,1)}{3^2} = \frac{6}{1} + \frac{11}{4} + \frac{6}{9} = \frac{113}{12}.$$

Obviously, the same result can be obtained via direct (and tedious) calculation. Table 1. gathers all divisors of n and their corresponding values of ω.

Table 1. Terms corresponding to all divisors d or 360, ordered in decreasing order of the vector of powers of consecutive primes

d	360	72	120	24	40	8	180	36	60	12	20	4
$\omega(d)$	3	2	3	2	2	1	3	2	3	2	2	1

d	90	18	30	6	10	2	45	9	15	3	5	1
$\omega(d)$	3	2	3	2	2	1	2	1	2	1	1	0

3.2. Sum of inverses

Let us now prove the following formula.

Theorem 3.4. *If n is any positive integer number, then*

$$\sum_{d|n} \frac{\binom{\omega(n)+1}{\omega(d)+1}}{S_{\omega(d)}(a)} x^{\omega(d)+1} = (1+x)^{\omega(n)+1} - 1. \tag{3.2}$$

In particular, if n is a square-free number, then

$$\sum_{d|n} \frac{x^{\omega(d)+1}}{\omega(d)+1} = \frac{(1+x)^{\omega(n)+1} - 1}{\omega(n)+1}. \tag{3.3}$$

Proof. We start by integrating equation (1.4) to obtain

$$\frac{(1+x)^{\omega(n)+1}}{\omega(n)+1} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)+1}}{\omega(d)+1} + C.$$

To find the integration constant, substitute $x = 0$ to find

$$C = \frac{1}{\omega(n)+1}.$$

Then,

$$\frac{(1+x)^{\omega(n)+1} - 1}{\omega(n)+1} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)+1}}{\omega(d)+1}$$

and after rearranging the terms,

$$\begin{aligned}(1+x)^{\omega(n)+1} - 1 &= \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{\omega(n)+1}{\omega(d)+1} x^{\omega(d)+1} \\ &= \sum_{d|n} \frac{\binom{\omega(n)+1}{\omega(d)+1}}{S_{\omega(d)}(a)} x^{\omega(d)+1},\end{aligned}$$

hence (3.2) holds. The equation (3.3) is derived in the same way. □

Example 3.5. If we substitute $x = 1$ in (3.2), we obtain

$$\sum_{d|n} \frac{\binom{\omega(n)+1}{\omega(d)+1}}{S_{\omega(d)}(a)} = 2^{\omega(n)+1} - 1.$$

In particular, for a square-free number n the above identity (or the equation (3.3)) can be expressed as

$$\sum_{d|n} \frac{1}{\omega(d)+1} = \frac{2^{\omega(n)+1}-1}{\omega(n)+1}.$$

Substitution $x = -1$ in (3.2) yields

$$\sum_{d|n} \frac{\binom{\omega(n)+1}{\omega(d)+1}}{S_{\omega(d)}(a)}(-1)^{\omega(d)} = 1.$$

The corresponding square-free case gives the following formula

$$\sum_{d|n} \frac{(-1)^{\omega(d)}}{\omega(d)+1} = \frac{1}{\omega(n)+1}.$$

Theorem 3.6. *Let n be any positive integer. Then we have the following formula*

$$\sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{(1+x)^k - 1}{k}. \tag{3.4}$$

In particular, if n is a square-free number, then

$$\sum_{d|n,d>1} \frac{x^{\omega(d)}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{(1+x)^k - 1}{k}. \tag{3.5}$$

Proof. We start by diving both sides of (1.4) by x. Then

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)-1} = \frac{(1+x)^{\omega(n)}}{x}.$$

We now integrate both sides of the above equality (the integration constant is included in left-hand-side only). The left-hand-side becomes

$$\int \left[\frac{1}{x} + \sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)-1}\right] dx = \ln|x| + \sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)} + C.$$

The right-hand-side now becomes:

$$\int \frac{(1+x)^{\omega(n)}}{x}\, dx = \sum_{k=1}^{\omega(n)} \frac{(x+1)^k}{k} + \ln|x|. \tag{3.6}$$

Comparing left-hand-side and right-hand-side we have

$$\sum_{k=1}^{\omega(n)} \frac{(x+1)^k}{k} = \sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)} + C.$$

To find the constant C, note that both sides are well-defined for $x = 0$, hence such a substitution implies

$$C = \sum_{k=1}^{\omega(n)} \frac{1}{k}.$$

Thus

$$\sum_{k=1}^{\omega(n)} \frac{(x+1)^k - 1}{k} = \sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)}$$

and (3.4) follows. The equation (3.5) is derived in the same way. □

Corollary 3.7. *The following identities hold for any square-free number n:*

$$\sum_{d|n,d>1} \frac{1}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{2^k-1}{k}, \tag{3.7}$$

$$\sum_{d|n,d>1} \frac{(-1)^{\omega(d)+1}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{1}{k}, \tag{3.8}$$

$$\sum_{\substack{d|n,d>1,\\ 2|\omega(d)}} \frac{2}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{2^k-2}{k}, \tag{3.9}$$

$$\sum_{\substack{d|n, d>1, \\ 2\nmid\omega(d)}} \frac{2}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{2^k}{k}. \tag{3.10}$$

PROOF. Identity (3.7) follows from (3.5) after taking $x = 1$. If we instead substitute $x = -1$, we obtain (3.8). To obtain (3.9), we subtract (3.8) from (3.7) and note that $1-(-1)^{\omega(d)+1} = 0$ if $\omega(d)$ is odd. If we now add (3.7) and (3.8), we obtain (3.10) (here, $1+(-1)^{\omega(d)+1} = 0$ if $\omega(d)$ is even). □

Theorem 3.6 can be generalized to arbitrary positive power of $\omega(d)$. Denote

$$\begin{aligned} h_k^{(1)} &:= 1, \quad k = 1, \ldots, \omega(n), \\ h_k^{(m)} &:= \sum_{j=k}^{\omega(n)} \frac{h_j^{(m-1)}}{j}, \quad k = 1, \ldots, \omega(n), \quad m > 1. \end{aligned}$$

THEOREM 3.8. *Let n and m be any positive integers. Then we have the following formula*

$$\sum_{d|n, d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)^m} = \sum_{k=1}^{\omega(n)} \left[\frac{(1+x)^k - 1}{k} \cdot h_k^{(m)} \right]. \tag{3.11}$$

If n is a square-free number, then

$$\sum_{d|n, d>1} \frac{x^{\omega(d)}}{\omega(d)^m} = \sum_{k=1}^{\omega(n)} \left[\frac{(1+x)^k - 1}{k} \cdot h_k^{(m)} \right]. \tag{3.12}$$

PROOF. The case $m = 1$ is covered by Theorem 3.6. The remaining cases follow from the induction principle.

Let $m \geq 1$ and suppose that (3.11) holds for m. Then dividing the formula (3.11) by x we obtain

$$\sum_{d|n, d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)-1}}{\omega(d)^m} = \sum_{k=1}^{\omega(n)} \left[\frac{(1+x)^k - 1}{kx} \cdot h_k^{(m)} \right].$$

Integrating both sides with respect to x gives the left-hand-side equal to

$$\sum_{d|n, d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)^{m+1}} + C,$$

while the right-hand-side, with the help of (3.6), equals

$$
\begin{aligned}
\int \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^k - 1}{kx} \cdot h_k^{(m)}\right] dx &= \sum_{k=1}^{\omega(n)} \int \frac{(x+1)^k}{kx} \cdot h_k^{(m)} dx - \sum_{k=1}^{\omega(n)} \int \frac{h_k^{(m)}}{kx} dx \\
&= \sum_{k=1}^{\omega(n)} \frac{h_k^{(m)}}{k} \left[\sum_{l=1}^{m} \frac{(x+1)^l}{l} + \ln|x|\right] - \sum_{k=1}^{\omega(n)} \frac{h_k^{(m)}}{k} \ln|x| \\
&= \sum_{k=1}^{\omega(n)} \frac{h_k^{(m)}}{k} \left[\sum_{l=1}^{k} \frac{(x+1)^l}{l}\right] \\
&= \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^k}{k} \cdot \sum_{j=k}^{\omega(n)} \frac{h_j^{(m)}}{j}\right] \\
&= \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^k}{k} \cdot h_k^{(m+1)}\right].
\end{aligned}
$$

Thus,

$$
\sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)^{m+1}} + C = \sum_{m=1}^{\omega(n)} \left[\frac{(x+1)^k}{k} \cdot h_k^{(m+1)}\right]
$$

and setting $x = 0$ yields

$$
C = \sum_{k=1}^{\omega(n)} \frac{1}{k} \cdot h_k^{(m+1)}.
$$

This in turn implies that

$$
\sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)^{m+1}} = \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^k - 1}{k} \cdot h_k^{(m+1)}\right],
$$

which completes the induction and proves (3.11). The formula (3.12) is derived in the same manner. □

Substitution $x = 1$ to the just obtained formulas yields the following identities.

Corollary 3.9. *Let n and m be any positive integers. Then*

$$\sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{1}{\omega(d)^m} = \sum_{k=1}^{\omega(n)} \left[\frac{2^k-1}{k} \cdot h_k^{(m)}\right].$$

If n is a square-free number, then

$$\sum_{d|n,d>1} \frac{1}{\omega(d)^m} = \sum_{k=1}^{\omega(n)} \left[\frac{2^k-1}{k} \cdot h_k^{(m)}\right].$$

Example 3.10. Let $n = 120 = 2^3 \cdot 3 \cdot 5$. Table 2 collects all the necessary calculation to find the corresponding sums in the case $m = 2$ and $m = 3$.

Table 2. Computation for $n = 120$

k	1	2	3
$h_k^{(2)}$	$\frac{11}{6}$	$\frac{5}{6}$	$\frac{1}{3}$
$h_k^{(3)}$	$\frac{85}{36}$	$\frac{19}{36}$	$\frac{1}{9}$
$\frac{2^k-1}{k}$	1	$\frac{3}{2}$	$\frac{7}{3}$
row2 · row4	$\frac{11}{6}$	$\frac{5}{4}$	$\frac{7}{9}$
row3 · row4	$\frac{85}{36}$	$\frac{19}{24}$	$\frac{7}{27}$

Thus,

$$\sum_{d|120,d>1} \frac{\binom{3}{\omega(d)}}{S_{\omega(d)}(3,1,1)} \frac{1}{\omega(d)^2} = \frac{11}{6} + \frac{5}{4} + \frac{7}{9} = \frac{139}{36}$$

and

$$\sum_{d|120,d>1} \frac{\binom{3}{\omega(d)}}{S_{\omega(d)}(3,1,1)} \frac{1}{\omega(d)^3} = \frac{85}{36} + \frac{19}{24} + \frac{7}{27} = \frac{737}{216}.$$

Note that in particular, for a square-free number n we obtain the following simple formula for squares of reciprocals.

$$\sum_{d|n,d>1} \frac{1}{\omega(d)^2} = \sum_{k=1}^{\omega(n)} \left[\frac{2^k-1}{k} \cdot \sum_{j=k}^{\omega(n)} \frac{1}{j}\right].$$

In the next result we will focus on finding the formula for the sum of reciprocals of rising factorials which arguments are the values of the prime omega function.

THEOREM 3.11. *For any positive integers n and m we have the following formula*

$$\sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)+m-1}}{\omega(d)^{\overline{m}}} = \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^{k+m-1}}{k^{\overline{m}}} - \sum_{j=0}^{m-1} \frac{x^j}{j!k^{\overline{m-j}}} \right]. \tag{3.13}$$

In particular, for any square-free number n we have

$$\sum_{d|n,d>1} \frac{x^{\omega(d)+m-1}}{\omega(d)^{\overline{m}}} = \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^{k+m-1}}{k^{\overline{m}}} - \sum_{j=0}^{m-1} \frac{x^j}{j!k^{\overline{m-j}}} \right]. \tag{3.14}$$

PROOF. The case $m = 1$ simplifies the equation (3.13) to

$$\sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^k}{k} - \frac{1}{k} \right],$$

which is the formula obtained in Theorem 3.6.

The general formula is obtained by induction. Suppose (3.13) holds for some $m \geq 1$. Then integrating that formula with respect to x we obtain

$$\int \sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)+m-1}}{\omega(d)^{\overline{m}}} dx = \sum_{d|n,d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)+m}}{\omega(d)^{\overline{m+1}}} + C$$

and

$$\int \left\{ \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^{k+m-1}}{k^{\overline{m}}} - \sum_{j=0}^{m-1} \frac{x^j}{j!k^{\overline{m-j}}} \right] \right\} dx$$

$$= \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^{k+m}}{k^{\overline{m+1}}} - \sum_{j=0}^{m-1} \frac{x^{j+1}}{(j+1)!k^{\overline{m-j}}} \right].$$

Substituting $x = 0$ we find

$$C = \sum_{k=1}^{\omega(n)} \frac{1}{k^{\overline{m+1}}}.$$

Thus, after shifting the range of summation,

$$\sum_{d|n, d>1} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \frac{x^{\omega(d)+m}}{\omega(d)^{\overline{m+1}}} = \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^{k+m}}{k^{\overline{m+1}}} - \sum_{j=1}^{m} \frac{x^j}{j! k^{\overline{m+1-j}}} \right] - \sum_{k=1}^{\omega(n)} \frac{1}{k^{\overline{m+1}}}$$

$$= \sum_{k=1}^{\omega(n)} \left[\frac{(x+1)^{k+m}}{k^{\overline{m+1}}} - \sum_{j=0}^{m} \frac{x^j}{j! k^{\overline{m+1-j}}} \right]$$

and (3.13) holds. The formula in (3.14) is obtained in the same way. □

3.3. Sum of powers

We begin this section with a variation of equation (3.1) in the case $m = 1$.

THEOREM 3.12. *For any positive integer n the following holds:*

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d) x^{\omega(d)} = \omega(n)(1+x)^{\omega(n)-1} x. \tag{3.15}$$

PROOF. To show the equality we multiply equation (1.4) by x:

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)+1} = (1+x)^{\omega(n)} x.$$

Now, we differentiate the above with respect to x:

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} (\omega(d)+1) x^{\omega(d)} = \omega(n)(1+x)^{\omega(n)-1} x + (1+x)^{\omega(n)}.$$

Subtracting (1.4) from the above equality we obtain (3.15). □

Recall that Stirling numbers of the second kind $S(m,k)$ with $m \geq 0$ and $0 \leq k \leq m$ are defined as follows:

$$S(m,0) = 0, S(m,m) = 1$$

for each $m \geq 0$ and

$$S(m+1,k) = k \cdot S(m,k) + S(m,k-1)$$

for each $m \geq 0$ and $1 \leq k \leq m$. In particular, we also have $S(m,1) = 1$ for each $m \geq 1$.

THEOREM 3.13. *If n is any positive integer and $1 \leq m \leq \omega(n)$, then*

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^m x^{\omega(d)} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k. \tag{3.16}$$

PROOF. The proof goes by induction on m. The case $m = 1$ is solved in Theorem 3.12.

Suppose the formula is true for some $m \geq 1$. Then

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^m x^{\omega(d)+1} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^{k+1}$$

and differentiating with respect to x we obtain

$$\begin{aligned}\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)}(\omega(d)^{m+1}+\omega(d)^m) &= \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k+1}}(1+x)^{\omega(n)-k-1}x^{k+1} \\ &\quad + \sum_{k=1}^{m}(k+1)S(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k.\end{aligned}$$

Then, subtracting (3.16) from the above and using the recursion of $S(m,k)$ we obtain

$$\begin{aligned}\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)}\omega(d)^{m+1} &= \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k+1}}(1+x)^{\omega(n)-k-1}x^{k+1} \\ &\quad + \sum_{k=1}^{m} kS(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k\end{aligned}$$

$$\begin{aligned}
&= S(m,m)\omega(n)^{\underline{m+1}}(1+x)^{\omega(n)-(m+1)}x^{m+1} \\
&\quad + S(m,1)\omega(n)^{\underline{1}}(1+x)^{\omega(n)-1}x \\
&\quad + \sum_{k=2}^{m} S(m,k-1)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k \\
&\quad + \sum_{k=2}^{m} kS(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k \\
&= S(m+1,m+1)\omega(n)^{\underline{m+1}}(1+x)^{\omega(n)-(m+1)}x^{m+1} \\
&\quad + S(m+1,1)\omega(n)^{\underline{1}}(1+x)^{\omega(n)-1}x \\
&\quad + \sum_{k=2}^{m} S(m+1,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k \\
&= \sum_{k=1}^{m+1} S(m+1,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k.
\end{aligned}$$

This completes the induction step and proves formula (3.16). $\square$

The next formula shows how to utilize falling factorials on both sides of the equation.

THEOREM 3.14. *If n is any positive integer and $1 \le m \le \omega(n)$, then*

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)}\omega(d)^{\underline{m}}x^{\omega(d)} = \omega(n)^{\underline{m}}(1+x)^{\omega(n)-m}x^m. \tag{3.17}$$

PROOF. Theorem 3.13 implies that

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)}x^{\omega(d)}\omega(d) = \omega(n)^{\underline{1}}(1+x)^{\omega(n)-1}x,$$

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)}x^{\omega(d)}\omega(d)^2 = \omega(n)^{\underline{2}}(1+x)^{\omega(n)-2}x^2 + \omega(n)(1+x)^{\omega(n)-1}x,$$

hence

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)} \omega(d)^{\underline{2}} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} x^{\omega(d)} (\omega(d)^2 - \omega(d))$$
$$= \omega(n)^{\underline{2}} (1+x)^{\omega(n)-2} x^2$$

and (3.17) holds for $m \in \{1, 2\}$.

The general formula follows from the induction. Suppose the theorem holds for some $m < \omega(n)$. Then,

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} x^{\omega(d)+1} = \omega(n)^{\underline{m}} (1+x)^{\omega(n)-m} x^{m+1}.$$

Differentiating with respect to x we get

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} (\omega(d)+1) x^{\omega(d)} = \omega(n)^{\underline{m+1}} (1+x)^{\omega(n)-(m+1)} x^{m+1}$$
$$+ (m+1)\omega(n)^{\underline{m}} (1+x)^{\omega(n)-m} x^m.$$

Then, by the induction hypothesis,

$$\begin{aligned} \omega(n)^{\underline{m+1}} (1+x)^{\omega(n)-(m+1)} x^{m+1} &= \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} (\omega(d)+1) x^{\omega(d)} \\ &\quad - (m+1)\omega(n)^{\underline{m}} (1+x)^{\omega(n)-m} x^m \\ &= \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} (\omega(d)+1) x^{\omega(d)} \\ &\quad - (m+1) \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} x^{\omega(d)} \\ &= \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} (\omega(d)-m) x^{\omega(d)} \\ &= \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m+1}} x^{\omega(d)}. \end{aligned}$$

Thus the theorem follows. □

Remark 3.15. Note that in Theorem 3.14 the left-hand-side of the stated formula has non-zero terms only if $\omega(d) \geq m$, otherwise, $\omega(d)^{\underline{m}} = 0$.

Recall that Lah numbers are the numbers $L(m,k)$ with $m \geq 0$ and $1 \leq k \leq m$ defined by the rule:

$$L(m,1) = m!, \quad L(m,m) = 1$$

for each $m \geq 1$ and

$$L(m,k) = (m+k-1) \cdot L(m-1,k) + L(m-1,k-1)$$

for each $m \geq 2$ and $1 \leq k \leq m$.

Remark 3.16. The sequence $L(m,k)$ has an explicit formula

$$L(m,k) = \frac{(m-1)!}{(k-1)!}\binom{m}{k}.$$

See [7] for further details. What interesting for us is that the numbers $L(m,k)$ appear as coefficients when expanding the rising factorials in the basis of falling factorials.

We now prove the numbers $L(m,k)$ also appear in the formula involving the terms $\omega(d)^{\overline{m}}$. This is also a surprising analogy – the same coefficients appear in a different scenario.

Theorem 3.17. *If n is a positive integer and $1 \leq m \leq \omega(n)$, then*

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\overline{m}} x^{\omega(d)} = \sum_{k=1}^{m} L(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k.$$

Proof. The proof is similar to the proof of Theorem 3.14. □

A simple series of identities can be derived with substitution $x = 1$.

Corollary 3.18. *For any positive integer n and any $1 \leq m \leq \omega(n)$ the following identities hold:*

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^m = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}} 2^{\omega(n)-k},$$

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} = 2^{\omega(n)-m}\omega(n)^{\underline{m}},$$

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\overline{m}} = \sum_{k=1}^{m} 2^{\omega(n)-k} L(m,k)\omega(n)^{\underline{k}}.$$

If we now suppose that n is square-free, the general formulas obtained in this section simplify. We cover that in the following theorem.

THEOREM 3.19. *Suppose that $n > 0$ is square-free and $1 \le m \le \omega(n)$. Then*

$$\sum_{d|n} \omega(d)^m x^{\omega(d)} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k, \tag{3.18}$$

$$\sum_{d|n} \omega(d)^{\underline{m}} x^{\omega(d)} = \omega(n)^{\underline{m}}(1+x)^{\omega(n)-m}x^m, \tag{3.19}$$

$$\sum_{d|n} \omega(d)^{\overline{m}} x^{\omega(d)} = \sum_{k=1}^{m} L(m,k)\omega(n)^{\underline{k}}(1+x)^{\omega(n)-k}x^k. \tag{3.20}$$

COROLLARY 3.20. *Suppose that $n > 0$ is square-free and $1 \le m \le \omega(n)$. Then*

$$\sum_{d|n} \omega(d)^m = \sum_{k=1}^{m} 2^{\omega(n)-k} S(m,k)\omega(n)^{\underline{k}}, \tag{3.21}$$

$$\sum_{d|n} \omega(d)^{\underline{m}} = 2^{\omega(n)-m}\omega(n)^{\underline{m}}, \tag{3.22}$$

$$\sum_{d|n} \omega(d)^{\overline{m}} = \sum_{k=1}^{m} 2^{\omega(n)-k} L(m,k)\omega(n)^{\underline{k}}. \tag{3.23}$$

In particular,

$$\sum_{k=1}^{m} 2^{\omega(n)-k} S(m,k)\omega(n)^{\underline{k}} = \sum_{k=1}^{\omega(n)} k^m S_k(a) = \sum_{k=1}^{\omega(n)} k^m \binom{\omega(n)}{k}. \tag{3.24}$$

PROOF. Identities (3.21)–(3.23) follow from substitution $x = 1$ in Theorem 3.19. Identity (3.24) follows from combining (3.21) and Theorem 3.2 and noting that $S_k(a) = \binom{\omega(n)}{k}$ for any square-free number n. $\square$

3.4. Example substitutions

A rather surprising identity holds when substituting $x = -1$ into any of the formulas (3.18)–(3.20).

Corollary 3.21. *For any positive integer n and $1 \le m \le \omega(n)$ we have*

$$\sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^m (-1)^{\omega(d)} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\overline{m}} (-1)^{\omega(d)} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)} \omega(d)^{\underline{m}} (-1)^{\omega(d)} = 0.$$

In particular, if n is square-free, then

$$\sum_{d|n} \omega(d)^m (-1)^{\omega(d)} = \sum_{d|n} \omega(d)^{\underline{m}} (-1)^{\omega(d)} = \sum_{d|n} \omega(d)^{\overline{m}} (-1)^{\omega(d)} = 0.$$

Corollary 3.22. *Suppose that $n > 0$ is square-free and $1 \le m \le \omega(n)$. Then*

$$\sum_{d|n} \omega(d)^{\underline{m}} (-1)^{\omega(d)} 2^{\omega(n)-\omega(d)} = \sum_{k=1}^{m} (-1)^k S(m,k) \omega(n)^{\underline{k}}, \tag{3.25}$$

$$(-1)^m \sum_{d|n} \omega(d)^{\underline{m}} 2^{\omega(n)-\omega(d)} (-1)^{\omega(d)} = \omega(n)^{\underline{m}}, \tag{3.26}$$

$$\sum_{d|n} \omega(d)^{\overline{m}} (-1)^{\omega(d)} 2^{\omega(n)-\omega(d)} = \sum_{k=1}^{m} (-1)^k L(m,k) \omega(n)^{\underline{k}}. \tag{3.27}$$

Proof. Identities (3.25)–(3.27) follow from substitution $x = -1/2$ in Theorem 3.19. For example, substitution $x = -1/2$ in (3.19) yields

$$\sum_{d|n} \omega(d)^{\underline{m}} \left(-\frac{1}{2}\right)^{\omega(d)} = \omega(n)^{\underline{m}} \left(\frac{1}{2}\right)^{\omega(n)-m} \left(-\frac{1}{2}\right)^{m}$$

which is equivalent to

$$\omega(n)^{\underline{m}} = (-1)^m \sum_{d|n} (-1)^{\omega(d)} \omega(d)^{\underline{m}} 2^{\omega(n)-\omega(d)}$$

and shows that (3.26) holds. The remaining identities are obtained in a similar way. $\square$

4. Identities involving Fibonacci and Lucas numbers

The next identities are in relation to the golden and silver means and Fibonacci numbers. Recall that $(F_r)_{r\geq 0}$ stands for classic Fibonacci numbers with $F_0 = 0$ and $F_1 = 1$ and $(L_r)_{r\geq 0}$ stands for Lucas numbers with $L_0 = 2$ and $L_1 = 1$.

Recall the following property of the golden mean.

LEMMA 4.1. *If* $\phi = \frac{\sqrt{5}+1}{2}$ *is the golden mean, then*

$$\phi^r = \frac{L_r}{2} + \frac{F_r}{2}\sqrt{5}$$

for any integer $r \geq 0$.

A similar, although less famous property holds for the silver mean.

LEMMA 4.2. *Suppose* $\Phi = \frac{\sqrt{5}-1}{2}$ *is the silver mean. Then*

$$\Phi^r = (-1)^r \left(\frac{L_r}{2} - \frac{F_r}{2}\sqrt{5}\right)$$

for any integer $r \geq 0$.

PROOF. Cases $r \in \{0, 1\}$ are immediate. The general case follows from the simple induction and uses the identities $L_r + F_r = 2F_{r+1}$ and $L_r + 5F_r = 2L_{r+1}$. □

LEMMA 4.3. *For any integers* $r \geq 0$ *and* $s \geq 0$ *we have the following:*

$$\phi^r \Phi^s = \frac{(-1)^s}{4} \left(L_r L_s - 5F_r F_s + \sqrt{5}(F_r L_s - F_s L_r)\right).$$

PROOF. Apply Lemma 4.1 and Lemma 4.2. □

We now derive several identities involving Fibonacci and Lucas numbers. They are up to our best knowledge new identities. Note that in the literature there are some identities combining Fibonacci, Lucas and Stirling numbers [1, 6].

Corollary 4.4. *Suppose n is a square-free number and $1 \leq m \leq \omega(n)$. Then*

$$\sum_{d|n} \omega(d)^m L_{\omega(d)} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}} L_{2\omega(n)-k}, \tag{4.1}$$

$$\sum_{d|n} \omega(d)^{\underline{m}} L_{\omega(d)} = \omega(n)^{\underline{m}} L_{2\omega(n)-m},$$

$$\sum_{d|n} \omega(d)^{\overline{m}} L_{\omega(d)} = \sum_{k=1}^{m} L(m,k)\omega(n)^{\underline{k}} L_{2\omega(n)-k},$$

$$\sum_{d|n} \omega(d)^m F_{\omega(d)} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}} F_{2\omega(n)-k}, \tag{4.2}$$

$$\sum_{d|n} \omega(d)^{\underline{m}} F_{\omega(d)} = \omega(n)^{\underline{m}} F_{2\omega(n)-m},$$

$$\sum_{d|n} \omega(d)^{\overline{m}} F_{\omega(d)} = \sum_{k=1}^{m} F(m,k)\omega(n)^{\underline{k}} L_{2\omega(n)-k}.$$

Proof. We only show identities (4.1) and (4.2) (the remaining identities can be proved in the same way). Substituting $x = \phi$ in (3.18) and using $1 + \phi = \phi^2$ we obtain

$$\sum_{d|n} \omega(d)^m \phi^{\omega(d)} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}} \phi^{2\omega(n)-k}.$$

Then, by Lemma 4.1 the above formula can be explicitly rewritten to

$$\sum_{d|n} \omega(d)^m \left(\frac{L_{\omega(d)}}{2} + \frac{F_{\omega(d)}}{2}\sqrt{5}\right) = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}} \left(\frac{L_{2\omega(n)-k}}{2} + \frac{F_{2\omega(n)-k}}{2}\sqrt{5}\right).$$

Note that both sides represent the element of the field $\mathbb{Q}(\sqrt{5})$, so

$$\sum_{d|n} \omega(d)^m L_{\omega(d)} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}} L_{2\omega(n)-k}$$

and

$$\sum_{d|n} \omega(d)^m F_{\omega(d)} = \sum_{k=1}^{m} S(m,k)\omega(n)^{\underline{k}} F_{2\omega(n)-k}. \qquad \square$$

Corollary 4.5. *Suppose n is a square-free number and $1 \le m \le \omega(n)$. Then*

$$2\sum_{d|n}\omega(d)^m(-1)^{\omega(d)}L_{\omega(d)}=\sum_{k=1}^{m}S(m,k)\omega(n)^{\underline{k}}(-1)^k(L_{\omega(n)-k}L_k-5F_{\omega(n)-k}F_k),$$

$$2\sum_{d|n}\omega(d)^{\underline{m}}(-1)^{\omega(d)}L_{\omega(d)}=\omega(n)^{\underline{m}}(-1)^m(L_{\omega(n)-m}L_m-5F_{\omega(n)-m}F_m),$$

$$2\sum_{d|n}\omega(d)^{\overline{m}}(-1)^{\omega(d)}L_{\omega(d)}=\sum_{k=1}^{m}L(m,k)\omega(n)^{\underline{k}}(-1)^k(L_{\omega(n)-k}L_k-5F_{\omega(n)-k}F_k),$$

$$2\sum_{d|n}\omega(d)^m(-1)^{\omega(d)}F_{\omega(d)}=\sum_{k=1}^{m}S(m,k)\omega(n)^{\underline{k}}(-1)^k(F_{\omega(n)-k}L_k-F_kL_{\omega(n)-k}),$$

$$2\sum_{d|n}\omega(d)^{\underline{m}}(-1)^{\omega(d)}F_{\omega(d)}=\omega(n)^{\underline{m}}(-1)^m(F_{\omega(n)-m}L_m-F_mL_{\omega(n)-m}),$$

$$2\sum_{d|n}\omega(d)^{\overline{m}}(-1)^{\omega(d)}F_{\omega(d)}=\sum_{k=1}^{m}L(m,k)\omega(n)^{\underline{k}}(-1)^k(F_{\omega(n)-k}L_k-F_kL_{\omega(n)-k}).$$

Proof. The proof is similar to the proof of Corollary 4.4. Here, we substitute $x = \Phi$ to the formulas in Theorem 3.19, use the fact that $1 + \Phi = \phi$ and apply Lemma 4.3. □

Remark 4.6. Note that to prove Corollary 4.5 we use the identity $(1+\Phi)^{\omega(n)-k}\Phi^k = \phi^{\omega(n)-k}\Phi^k$. We can also use the following identity

$$(1+\Phi)^{\omega(n)-k}\Phi^k = \phi^{\omega(n)}\Phi^{2k}$$

to obtain slightly different right-hand-side of the identities in Corollary 4.5, i.e., the terms

$$L_{\omega(n)}L_{2k} - 5F_{\omega(n)}F_{2k} \quad \text{and} \quad F_{\omega(n)}L_{2k} - F_{2k}L_{\omega(n)}$$

and the term $(-1)^k$ in any of these identities changes to $(-1)^{2k} = 1$.

We now show the application of the equations involving reciprocals.

Corollary 4.7. *For any square-free number n we have the following identities:*

$$\sum_{d|n}\frac{L_{\omega(d)+1}}{\omega(d)+1} = \frac{L_{2\omega(n)+2}-1}{\omega(n)+1}, \tag{4.3}$$

$$\sum_{d|n} \frac{F_{\omega(d)+1}}{\omega(d)+1} = \frac{F_{2\omega(n)+2}}{\omega(n)+1}, \tag{4.4}$$

$$\sum_{d|n} \frac{(-1)^{\omega(d)+1} L_{\omega(d)}}{\omega(d)+1} = \frac{L_{\omega(n)+1}-1}{\omega(n)+1}, \tag{4.5}$$

$$\sum_{d|n} \frac{(-1)^{\omega(d)} F_{\omega(d)}}{\omega(d)+1} = \frac{F_{\omega(n)+1}}{\omega(n)+1}, \tag{4.6}$$

$$\sum_{d|n, d>1} \frac{L_{\omega(d)}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{L_{2k}-1}{k}, \tag{4.7}$$

$$\sum_{d|n, d>1} \frac{F_{\omega(d)}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{F_{2k}}{k}, \tag{4.8}$$

$$\sum_{d|n, d>1} \frac{(-1)^{\omega(d)} L_{\omega(d)}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{L_k-1}{k}, \tag{4.9}$$

$$\sum_{d|n, d>1} \frac{(-1)^{\omega(d)+1} F_{\omega(d)}}{\omega(d)} = \sum_{k=1}^{\omega(n)} \frac{F_k}{k}. \tag{4.10}$$

Proof. The proof is similar to the proof of previous identities obtained in this section. Here, we substitute $x = \phi$ and $x = \Phi$ in the equation (3.3) and use $1 + \phi = \phi^2$, $1 + \Phi = \phi$, to obtain identities (4.3)–(4.6). The same substitutions but in the equation (3.5) lead to identities (4.7)–(4.10). □

Remark 4.8. All formulas in this section are provided in the square-free case. If we include the coefficient $\frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a)}$ under the sum $\sum\limits_{d|n} \cdots$, then the corresponding formula represents the identity for arbitrary positive integer n.

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Institute of Mathematics
Pedagogical University of Kraków
Podchorążych 2
30-084 Kraków
Poland
e-mail: karol.gryszka@up.krakow.pl